22 February 1999

ELSEVIER

Physics Letters A 252 (1999) 173–179

PHYSICS LETTERS A

Two hierarchies of ordinary differential equations and their properties

Nicolai A. Kudryashov

Department of Applied Mathematics, Moscow Engineering Physics Institute, 31 Kashirskoe Shosse, Moscow 115409, Russia

Received 28 September 1998; accepted for publication 7 December 1998
Communicated by A.R. Bishop

Abstract

Two hierarchies of ordinary differential equations are presented. Relations between them are given. Rational and special solutions of one hierarchy are found. © 1999 Elsevier Science B.V.

PACS: 02.30.+g; 03.40.kf
Keywords: Painlevé equations; Painlevé property; Transcendents

1. Introduction

In the recent papers [1–3] we have made an attempt to solve the problem of the definition of new functions except the Painlevé transcendents which are determined by nonlinear ordinary differential equations. There are a few approaches to look for the equations which have transcendents but we used the simplest one based on the application of the Painlevé conjecture by Ablowitz, Ramani and Segur [4,5]. Usually this conjecture is used as the Painlevé ODE test: if a given partial differential equation is reducible to an ordinary differential equation not of Painlevé type then the Painlevé ODE test predicts the partial differential equation is not complete integrable [4,5]. However, on the other hand, using the integrable partial differential equation by reduction one can obtain an ODE having the Painlevé property. Thus, one can look for ODEs that can have as solutions the transcendental functions with respect to constants of integration.

We applied this approach and obtained two hierarchies of ODEs which take the form [1]

$$d^{n+1}(u) = \frac{z}{2} \quad (n = 1, 2, \ldots)\,, \tag{1.1}$$

and

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} + 2v\right) d^n(v_z - v^2) - zv = \alpha\,, \tag{1.2}$$

where the operator d^n in Eqs. (1.1) and (1.2) is determined by the Lenard relation

$$\frac{\mathrm{d}}{\mathrm{d}z} d^{n+1} = d^n_{zzz} + 2ud^n_z + 2u_z d^n\,,$$
$$d^0 = \frac{1}{2}, \quad d^1 = u\,. \tag{1.3}$$

It should be noted that hierarchies (1.1) and (1.2) have remarkable properties.

Assuming $n = 1$, we obtain the first and second Painlevé equations,

$$u_{zz} + 3u^2 = \frac{z}{2}\,, \tag{1.4}$$

PII S0375-9601(98)00950-5

$$u_{zz} = 2v^3 + zv + \alpha , \tag{1.5}$$

from Eqs. (1.1) and (1.2).

If we take $n = 2$ in Eqs. (1.1) and (1.2), we will find the fourth order equation

$$u_{zzzz} + 5u_z^2 + 10uu_{zz} + 10u^3 = \frac{z}{2} \tag{1.5}$$

and

$$v_{zzzz} = 10v^2 v_{zz} + 10vv_z^2 - 6v^5 + zv + \alpha . \tag{1.6}$$

It is well known that Eqs. (1.4) and (1.5) determine new type of functions that are the Painlevé transcendents. The question arises as to whether there are new functions determined by Eqs. (1.5), (1.6).

We checked Eqs. (1.5) and (1.6) by the Painlevé test and observed these equations pass this [2]. There are also the Lax pairs for Eqs. (1.1) and (1.2) [2]. It turned out that solutions of Eqs. (1.5) and (1.6) are the essentially transcendental functions with respect to constants of integration [3].

The aim of this Letter is to introduce two other hierarchies and to show these ones have similar properties as Eqs. (1.1) and (1.2).

The outline of this work is as follows. Two hierarchies that we are going to study are introduced in Section 2. Results of the investigation of two fourth order equations in the Painlevé test are given in Section 3. The relations between these hierarchies are presented in Section 4 where we obtain some rational and special solutions of one fourth order equation.

2. Hierarchies studied

In order to introduce two integrable hierarchies of ODEs we use the Caudrey-Dodd-Gibbon and the Kaup-Kupershmidt equations which take the form [6–10]

$$u_t + \Theta_1 G_n(u) = 0 \quad (n = 1, 2, \dots) \tag{2.1}$$

and

$$a_t + \Theta_2 H_n(a) = 0 \quad (n = 1, 2, \dots) , \tag{2.2}$$

where operators Θ_1 and Θ_2 are determined by formulas

$$\Theta_1 = \Theta_2 = D^3 + 2uD + u_x , \quad D = \frac{\partial}{\partial x} \tag{2.3}$$

and operators G_n and H_n can be obtained by recursion relations [9]

$$G_{n+2} = J_1(u)\Theta_1(u)G_n \tag{2.4}$$

and

$$H_{n+2} = J_2(a)\Theta_2(a)H_n , \tag{2.5}$$

when

$$G_0 = 1 , \quad G_1 = u_{xx} + \tfrac{1}{4}u^2 , \tag{2.6}$$

$$H_0 = 1 , \quad H_1 = a_{xx} + 4a^2 . \tag{2.7}$$

Thus, operators J_1 and J_2 are determined by formulas

$$J_1 = D^3 + \tfrac{1}{2}D^2 uD^{-1} + \tfrac{1}{2}D^{-1}uD^2 + \tfrac{1}{8}(u^2 D^{-1} + D^{-1}u^2) ,$$

$$D^{-1} = \int \mathrm{d}x , \tag{2.8}$$

and

$$J_2 = D^3 + 3(uD + Du) + 2(D^2 uD^{-1} + D^{-1}uD^2) + 8(u^2 D^{-1} + D^{-1}u^2) . \tag{2.9}$$

The modified equations for Eqs. (2.1) and (2.2) can be presented in the form

$$W_t + D(D + W)G_n(W_x - \tfrac{1}{2}W^2) = 0 , \tag{2.10}$$

$$V_t + D(D + V)H_n(V_x - \tfrac{1}{2}V^2) = 0 . \tag{2.11}$$

Remark 1. One can note that Eqs. (2.10) and (2.11) coincide using the variables $W = 2V$ and consequently we cannot tell these equations apart.

When the Painlevé test is applied to Eqs. (2.1) and (2.2) then the singular manifold equations arise which can be written as follows [9,10],

$$Z_t + Z_x H_n(\{Z; x\}) = 0 , \tag{2.12}$$

and

$$\Phi_t + \Phi_x G_n(\{\Phi; x\}) = 0 , \tag{2.13}$$

where $\{Z; x\}$ is the Schwarzian derivative [9]

$$\{Z; x\} = \frac{Z_{xxx}}{Z_x} - \frac{3}{2}\frac{Z_{xx}^2}{Z_x^2} . \tag{2.14}$$

It is easy to see that Eqs. (2.1), (2.2), (2.10), (2.11), (2.12) and (2.13) admit the group delation and consequently the solutions of these equations can be looked for as self-similar ones. For example, if we take

$$Z(x,t) = t^{m_n/(2n+1)}\varphi(z)\,,\quad z = x[(2n+1)t]^{-1/(2n+1)}\,, \tag{2.15}$$

then we have the hierarchy of ODEs in the form

$$\varphi_z h_n(\{\varphi; z\}) - z\varphi_z + m_n\varphi = 0\,, \tag{2.16}$$

from Eqs. (2.12) where the operator h_n is determined by formulas (2.5), (2.7) and (2.9) corresponding to the abovementioned operator H_n, Θ_2 and J_2 at using $x \to z$.

By analogy, taking into account the self-similar variables

$$\Phi(x,t) = t^{P_n/(2n+1)}\Psi(z)\,,\quad z = x[(2n+1)t]^{-1/(2n+1)}\,, \tag{2.17}$$

one can obtain the following hierarchy of ODEs,

$$\Psi_z g_n(\{\Psi; z\}) - z\Psi_z + p_n\Psi = 0\,, \tag{2.18}$$

from Eq. (2.13). The operator g_n in (2.18) is determined by operators G_n, Θ_1 and J_1 at $x \to z$.

Assuming $m_n = 0$ in Eqs. (2.16) we obtain the hierarchy of ODEs,

$$h_n(F) = z\,, \tag{2.19}$$

if we take into account

$$F = \{\varphi; z\}\,. \tag{2.20}$$

We have the first Painlevé equation

$$F_{zz} + 4F^2 = z \tag{2.21}$$

from Eqs. (2.19) at $n = 1$.

One can obtain the fourth order equation

$$F_{zzzz} + 12FF_{zz} + 2F_z^2 + \tfrac{32}{3}F^3 = z\,, \tag{2.22}$$

from hierarchy (2.19) assuming $n = 2$.

Remark 2. Assuming $p_n = 0$ in Eqs. (2.18) also leads to Eq. (2.19) after the change of variables and consequently we cannot tell these equations apart either.

On the other hand, one can see that Eq. (2.22) differs from Eq. (1.5) considered earlier [1].

Under the action of the operator $(\varphi_z)^{-1}\mathrm{d}/\mathrm{d}z$, the sequence of ODEs (2.16) becomes

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} + \omega\right) h_n\left(\omega_z - \frac{1}{2}\omega^2\right) - z\omega + \beta_n = 0\,, \tag{2.23}$$

if we take into account

$$\omega = \frac{\varphi_{zz}}{\varphi_z}\,,\quad \beta_n = m_n - 1\,. \tag{2.24}$$

The hierarchy of ODEs (2.23) is also different from the sequence of ODEs (1.2) discussed in Ref. [1]. This correspond to the modified equation (2.11) if we also use the self-similar variables.

At $n = 1$ we also have the fourth order equation

$$\omega_{zzzz} + 5\omega_z\omega_{zz} - 5\omega^2\omega_{zz} - 5\omega\omega_z^2 + \omega^5 - z\omega + \beta_n = 0\,. \tag{2.25}$$

It should be noted that Eqs. (2.19) and (2.23) were obtained by reduction of integrable partial differential equations to ODEs and consequently one can expect that these sequences of ODEs are integrable ones because these equations have to possess the Painlevé property in accordance with the conjecture by Ablowitz, Ramani and Segur. Curiously, there are two sequences of ODEs (1.1) and (2.19) which give the first Painlevé equation at $n = 1$ but certainly these hierarchies are distinguished in the general case.

3. Painlevé test for Eqs. (2.22) and (2.25)

In this section we want to apply the Painlevé test to Eqs. (2.22) and (2.25) to determine the behavior of the movable critical points in their general solutions.

To study these equations we use the Painlevé test for ODEs presented in Refs. [4,11]. The essence of this method can be given in several ways.

Let the ordinary differential equation

$$E(u,z) = 0 \tag{3.1}$$

be given. We keep in mind Eqs. (2.22) and (2.25).

In the first place we look for all possible families of solutions of Eq. (3.1) assuming

$$u \simeq \frac{u_0}{(z-z_0)^p}\,, \tag{3.2}$$

where p is the order of the singularity, z_0 is a movable singularity and u_0 are constants which can be found after substitution of (3.2) into Eq. (3.1).

The first necessary condition for the absence of movable critical singularities is that all p are integers [11,12].

We take Eq. (2.22) in the following form,

$$F_{zzzz} + \tfrac{3}{2}FF_{zz} + \tfrac{3}{4}F_z^2 + \tfrac{1}{6}F^3 - z = 0. \qquad (3.3)$$

Substitution of (3.2) into Eq. (3.3) shows this equation admits two families of solutions with (p, u_0): $(-2, -12)$ and $(-2, -60)$.

Using Eq. (2.25) in the form

$$\omega_{zzzz} + \tfrac{5}{2}\omega_z\omega_{zz} - \tfrac{5}{4}\omega^2\omega_{zz} - \tfrac{5}{4}\omega\omega_z^2 + \tfrac{1}{16}\omega^5 - z\omega + \beta_n = 0, \qquad (3.4)$$

one can obtain this equation has four families of solutions (p, u_0): $(-1, -4)$, $(-1, 2)$, $(-1, 8)$ and $(-1, -6)$. It is evident that the first necessary condition for the integrability of Eqs. (3.3) and (3.4) holds.

The second necessary condition for the absence of movable critical points in the general solution of Eq. (3.1) is that the degree N of Eq. (3.1) is equal to the amount of the Fuchs indices and all N these indices of all families are different integers [10,11].

Substituting

$$u \simeq \frac{u_0}{(z - z_0)^p} + u_j(z - z_0)^{j-p} \qquad (3.5)$$

into Eq. (3.1) and equating the expression of the one power of u_j to zero we have the polynomial equation for definition of the Fuchs indices.

Substitution of (3.5) into Eq. (3.3) with $u_0 = -12$ and $p = -2$ gives an equation which has four different roots: $j_1 = 1$, $j_2 = 3$, $j_3 = 4$ and $j_4 = 8$.

Another family of solution of Eq. (3.3) with $u_0 = -60$ and $p = -2$ leads to an equation which gives the following Fuchs indices: $j_1 = -1$, $j_2 = -5$, $j_3 = 8$ and $j_4 = 12$.

One can see that the second necessary condition for the solution of Eq. (3.3) also holds.

We now consider the second condition for Eq. (3.4). Substituting (3.5) into Eq. (3.4) and equating of the expression of the one power of u_j to zero we obtain the same equation for the Fuchs indices at $u_0 = -4$ and at $u_0 = 2$. This equation has the following roots: $j_1 = -1$, $j_2 = 2$, $j_3 = 3$ and $j_4 = 6$.

In the case $u_0 = -6$ we obtain an equation which give the Fuchs indices: $j_1 = -1$, $j_2 = -2$, $j_3 = 6$ and $j_4 = 7$ and we have a polynomial equation at $u_0 = 8$ which has the roots: $j_1 = -1$, $j_2 = -7$, $j_3 = 6$ and $j_4 = 12$.

These results shows that the second necessary condition for the absence of movable critical points in the general solution of Eq. (3.4) holds as well.

The third necessary condition corresponds to checking the existence of the Laurent series for the solution of Eq. (3.1).

For the families with positive indices the algorithm is reduced to substitution of the expression

$$u = \sum_{i=0}^{\infty} u_i(z - z_0)^{i-p} \qquad (3.6)$$

into Eq. (3.1) and control that u_{j_k} (j_k are the Fuchs indices) can be taken as arbitrary constants.

For the family with positive indices of Eq. (3.3) we have obtained that z_0, a_3, a_4 and a_8 are the arbitrary constants in this expansion and the third necessary condition for the family of solution with positive indices holds.

For the families with positive indices of Eq. (3.4) we have also found that z_0, a_2, a_3 and a_6 are the arbitrary constants and consequently the third necessary condition for the solutions of Eq. (3.4) also holds. Consequently Eqs. (2.22) and (2.25) pass the Painlevé test for the families of solution with positive indices but we have not investigated the third necessary condition for the families of solutions with the negative indices because of the length of the calculations.

4. Rational and special solutions of Eq. (2.25)

In this section we consider the rational and special solutions of Eq. (2.25).

The modified equation for the Caudrey–Dodd–Gibbon and the Kaup–Kupershmidt equations is known to have two different singular manifold equations [13,14]. Using self-similar solutions (2.15) we obtain two relations for Eq. (2.25) and Eqs. (2.16) and (2.18). They can be written in the following form,

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}+\omega\right)h_n(\omega_z-\tfrac{1}{2}\omega^2)-z\omega+m_n-1$$
$$=\frac{1}{\varphi_z}\frac{\mathrm{d}}{\mathrm{d}z}[\varphi_z h_n(\{\varphi;z\})-z\varphi_z+m_n\varphi]\,, \qquad (4.1)$$

where

$$\omega=\frac{\varphi_{zz}}{\varphi_z} \qquad (4.2)$$

and

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}+\omega\right)h_n(\omega_z-\tfrac{1}{2}\omega^2)-z\omega-\tfrac{1}{2}p_n+\tfrac{1}{2}$$
$$=\frac{1}{2\Psi_z}\frac{\mathrm{d}}{\mathrm{d}z}[\Psi_z g_n(\{\Psi;z\})-z\Psi_z+p_n\Psi]\,, \qquad (4.3)$$

where

$$\omega=-\frac{\Psi_{zz}}{\Psi_z}\,. \qquad (4.4)$$

Taking into account the approach suggested in Ref. [14] for the derivation of the rational and special solutions for nonlinear partial differential equations one can obtain the iterative formulas of Weiss [9] which take the form

$$\Phi_{n+1,x}=\frac{Z_n^4}{Z_{n,x}^2} \qquad (4.5)$$

and

$$Z_{k+1,x}=\frac{\Phi_k}{\Phi_{k,x}^{1/2}}\,, \qquad (4.6)$$

where $Z(x,t)$ and $\Phi(x,t)$ are solutions of Eqs. (2.12) and (2.13).

Assuming that [15]

$$Z_n(x,t)=t^{m_n/(2n+1)}\varphi_n(z)\,,$$
$$z=x[(2n+1)t]^{-1/(2n+1)}\,, \qquad (4.7)$$
$$Z_{k+1}(x,t)=t^{M_n/(2n+1)}\varphi_n(z)\,, \qquad (4.8)$$
$$\Phi_k(x,t)=t^{p_k/(2k+1)}\Psi_k(z)\,,$$
$$z=x[(2n+1)t]^{-1/(2n+1)}\,, \qquad (4.9)$$
$$\Phi_{n+1}(x,t)=t^{P_n/(2n+1)}\Psi_{n+1}(z)\,. \qquad (4.10)$$

Substitutions of (4.7)–(4.10) into transformations (4.5) and (4.6) give formulas

$$\Psi_{n+1,z}=\frac{\varphi_n^4}{\varphi_{n,z}^2} \qquad (4.11)$$

and

$$\varphi_{k+1,z}=\frac{\Psi_k}{\Psi_{k,z}^{1/2}} \qquad (4.12)$$

at

$$M_k=\frac{p_k+3}{2}\,,\quad P_n=2m_n+3\,. \qquad (4.13)$$

Constants M_k and P_n determined by formulas (4.13) correspond to the new values of parameters in Eqs. (2.16) and (2.18).

Relations (4.1), (4.2) and formulas (4.11) and (4.12) can be used for deriving rational and special solutions of Eqs. (2.16) and (2.18) and consequently of Eqs. (2.23).

Without loss of generality let us find the rational and special solutions of Eq. (2.25) using Eqs. (2.16) and (2.18) at $n=1$. These equations take the form

$$\varphi_z[\{\varphi;z\}_{zz}+4\{\varphi;z\}^2]-z\varphi_z+m_n\varphi=0 \qquad (4.14)$$

and

$$\Psi_z[\{\Psi;z\}_{zz}+\tfrac{1}{4}\{\Psi;z\}^2]-z\Psi_z+p_n\Psi=0\,. \qquad (4.15)$$

Solutions of Eq. (2.25) can be obtained by formulas

$$\omega=\frac{\varphi_{zz}}{\varphi_z}\,,\quad \omega=-\frac{\Psi_{zz}}{2\Psi_z}\,. \qquad (4.16)$$

It is clear that we have the simplest solutions $\varphi_0=\Psi_0=z$ of Eqs. (4.14) and (4.15) at $m_n=p_n=1$ which corresponds to the trivial solution $\omega_0=0$ of Eq. (2.25) at $\beta_0=0$.

Using φ_0 and Ψ_0 one can find the rational solutions of Eqs. (4.14) and (4.15) from the iterative formulas

$$\Psi_{n+1}=\int\frac{\varphi_n^4}{\varphi_{n,z}^2}\,\mathrm{d}z\,,$$
$$\varphi_{k+1}=\int\frac{\Psi_k}{\Psi_{k,z}^{1/2}}\,\mathrm{d}z\,. \qquad (4.17)$$

We have

$$\varphi_1=\tfrac{1}{2}z^2\quad(m_n=2)\,,\quad \omega(z,1)=z^{-1}\,, \qquad (4.18)$$

and

$$\Psi_1=\tfrac{1}{5}(z^5+36)\quad(p_n=5)\,,$$
$$\omega(z,-2)=-2z^{-1}\,. \qquad (4.19)$$

Further, we obtain

$$\varphi_2 = \frac{z^5 - 144}{20z} \quad (m_n = 4)\,,$$

$$\omega(z,3) = \frac{3z^6 - 72}{z^5 + 36}\,, \tag{4.20}$$

and

$$\Psi_2 = \frac{1}{112} z^7 \quad (p_n = 7)\,,$$

$$\omega(z,-3) = -3z^{-1}\,. \tag{4.21}$$

By analogy one can also find

$$\varphi_3 = z^5 - 504 \quad (m_n = 5)\,,$$

$$\omega(z,4) = 4z^{-1}\,, \tag{4.22}$$

$$\Psi_3 = \frac{z^{11}}{70400} - \frac{27z^6}{1600} + \frac{10611}{400} z + \frac{3645z}{4(z^5+36)} \quad (p_n = 11)\,, \tag{4.23}$$

and

$$\Psi_4 = \tfrac{1}{325} z^{13} - \tfrac{252}{25} z^8 + \tfrac{508032}{25} z^3 + \tfrac{2256048128}{25} z^{-2} - \tfrac{9217732608}{35} z^{-7} \quad (p_n = 13)\,. \tag{4.24}$$

Substitutions of (4.23) and (4.24) into (4.16) give $\omega(z,-5)$ and $\omega(z,-6)$.

One can see from (4.13) that the rational solutions of Eq. (2.25) can be obtained for the all positive and negative integers of parameter β_n,

$$\beta_n = n\,, \quad n \neq 2 \pm 3k \quad (k = 0,1,2,...)\,. \tag{4.25}$$

Let now us consider the special solutions of Eq. (2.25). One can note that we have the first Painlevé equation,

$$F_{zz} + 4F^2 = z\,, \tag{4.26}$$

from Eqs. (4.14) and (4.15) at $m_n = p_n = 0$ if we assume

$$F = \omega_z - \tfrac{1}{2}\omega^2 = \{\varphi; z\}\,. \tag{4.27}$$

Therefore, at given solution of Eq. (4.26) one can find the special solution $\omega(z,\beta_n)$ from the Riccatti equation (4.27) and then one can obtain $\varphi(z)$ and $\Psi(z)$ from Eqs. (4.16). Using formulas (4.17) one can have the special solutions of Eq. (2.25) at other values of parameter β_n. These values of parameter β_n are determined by formulas

$$\beta_n = \tfrac{1}{2} \pm 3k \quad (k = 0,1,2,...) \tag{4.28}$$

and

$$\beta_n = 2 \pm 3k \quad (k = 0,1,2,...)\,. \tag{4.29}$$

Therefore, the rational and special solutions of Eq. (2.25) can be found for all positive and negative values of parameter β_n and at half-integers determined by formula (4.28).

5. Conclusion

We have presented two hierarchies of nonlinear ordinary differential equations and have considered some properties of these hierarchies. These hierarchies were obtained using the self-similar solutions of the nonlinear integrable partial differential equations and consequently by the conjecture of Ablowitz, Ramani and Segur these hierarchies of ODEs have to be integrable too. We have considered the application of the Painlevé test to the fourth order equations of hierarchies and obtained that the studied equations pass the Painlevé test. Using the Bäcklund transformations presented in this work for one of the fourth order equations of the hierarchy some rational and special solutions were obtained which exist for all the integers and for some half-integers values of the parameter of Eq. (2.25).

Acknowledgement

This work was supported by the International Science and Technology Center under project B23-96. This material is partially based upon work supported by the US Civilian Research and Development Foundation under award no. RG2-144.

References

[1] N.A. Kudryashov, Phys. Lett. A 224 (1997) 353.
[2] N.A. Kudryashov, M.B. Soukharev, Phys. Lett. A 237 (1998) 206.
[3] N.A. Kudryashov, J. Phys. A 31 (1998) L129.

[4] M.J. Ablowitz, A. Ramani, H. Segur, J. Math. Phys. 21 (1980) 715; 1006.
[5] M.J. Ablowitz, P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering (Cambridge Univ. Press, Cambridge, 1991).
[6] P.J. Caudrey, R.K. Dodd, J.D. Gibbon, Proc. R. Soc. A 351 (1976) 407.
[7] R.K. Dodd, J.D. Gibbon, Proc. R. Soc. A 358 (1977) 287.
[8] B Kuperschmidt, G. Wilson, Invent. Math. 62 (1981) 403.
[9] J. Weiss, J. Math. Phys. 25 (1984) 13.
[10] J. Weiss, J. Meth. Phys. 27 (1986) 1293.
[11] R. Conte, in: The Painlevé Property, One Century Later, R. Conte, ed., CRM Series in Mathematical Physics (Springer, Berlin, 1998).
[12] R. Conte, A.P. Fordy, A. Pickering, Physica D 69 (1993) 33.
[13] N.A. Kudryashov, J. Phys. A 27 (1994) 2457.
[14] N.A. Kudryashov, J. Phys. A 30 (1997) 5445.
[15] N.A. Kudryashov, A. Pickering, J. Phys. A., submitted.