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The second Painlevé equation as a model for the electric field in a semiconductor

Nicolai A. Kudryashov

Department of Applied Mathematics, Moscow Engineering Physics Institute, 31 Kashirskoe Shosse, Moscow 115409, Russian Federation

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Abstract

It is shown that the second Painlevé equation can be used as a model for describing the electric field in a semiconductor. The derivation of the asymptotic and special solutions of the second Painlevé equation is discussed in the framework of this model. © 1997 Published by Elsevier Science B.V.

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The second Painlevé equation was found by Painlevé [1, 2] who studied the class of second order ordinary differential equations (ODEs). Painlevé, Gambier and their pupils found 50 second order ODEs having a canonical form and whose solutions do not have any movable critical singularities, which is now called having the Painlevé property. Painlevé, Gambier and Fuchs also found six new functions which are defined by nonlinear ODEs depending on complex parameters. At the present time these functions are called Painlevé transcendents. One of these equations discovered is called the second Painlevé equation and has the form [1]

$$\frac{d^2y}{dz^2} = 2y^3 + zy + \alpha. \qquad (1)$$

The current interest in the Painlevé property stems from the observation by Ablowitz and Segur that reductions of partial differential equations of solution type gave rise to ODEs whose movable singularities were only poles [3-5]. This led to their famous conjecture: "All reductions of a completely integrable partial differential equation (PDE) are of Painlevé type perhaps only after a change of variables" and yielded later the definition of the Painlevé test for PDEs [6,7].

However, although the Painlevé equations were first found from strictly mathematical considerations, they have recently appeared in several physical applications [1]. The aim of this Letter is to show that Eq. (1) can also be used as a model for describing the electric field in a semiconductor. Let us take the set of equations for describing the dynamics of electrons and holes in a semiconductor [8-10],

$$n - p = k^2 \frac{\partial^2 \varphi}{\partial x^2}, \qquad (2)$$

$$\frac{\partial n}{\partial x} - n\frac{\partial \varphi}{\partial x} = I_n, \qquad (3)$$

$$\frac{\partial p}{\partial x} + p\frac{\partial \varphi}{\partial x} = I_p, \qquad (4)$$

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where n is the electron concentration, p is the hole concentration, φ is the potential of the electric field, k is Boltzmann's constant, $I_n = -j_n/\mu_n$ (j_n is the electron current density, μ_n is the electron mobility), $I_p = -j_p/\eta_p$ (j_p is the hole current density and η_p is the hole mobility correspondingly).

It is necessary to add some more equations in order to fully describe the processes in a semiconductor [8,9]

$$\frac{\partial n}{\partial t} + \frac{\partial j_n}{\partial x} = g - R_1(n,p), \tag{5}$$

$$\frac{\partial p}{\partial t} + \frac{\partial j_p}{\partial x} = g - R_2(p,n), \tag{6}$$

where g is the photogeneration rate of electron–hole pairs, and $R_1(n,p)$ and $R_2(p,n)$ are the recombination rates of electron–hole pairs.

The set of equations (2)–(6) is written down in dimensionless form as in Ref. [8]. This set was repeatedly used for the numerical modeling of semiconductor diodes. The problem which arose in this modeling was the small parameter k in Eq. (2). For example, for silicon transistors of typical size we have $k \sim 10^{-3}$–10^{-8} [8], which leads to a very small ($\sim k^2$) mesh width in time. There have been a number of attempts to overcome this difficulty (see, for example, Refs. [8,9]) but we will not discuss these difficulties here.

Let us show that the electric field in a semiconductor can be described by the second Painlevé equation (1) in the special case when j_n and j_p depend on t. Subtracting Eq. (4) from Eq. (3) gives the equality

$$k^2 \frac{\partial^3 \varphi}{\partial x^3} - (p+n)\frac{\partial \varphi}{\partial x} = I_n - I_p \tag{7}$$

if we take into account Eq. (2). Denoting

$$E = -\frac{\partial \varphi}{\partial x}$$

we obtain

$$n + p = \frac{1}{E}(I_n - I_p + k^2 E_{xx}), \tag{8}$$

where the subscript x denotes differentiation. Using Eqs. (2) and (8) we have

$$p = \frac{1}{2E}\left(k^2 E_{xx} + I_n - I_p\right) + \tfrac{1}{2}k^2 E_x, \tag{9}$$

$$n = \frac{1}{2E}\left(k^2 E_{xx} + I_n - I_p\right) - \tfrac{1}{2}k^2 E_x. \tag{10}$$

Substituting Eq. (9) into Eq. (4) leads to the following equation,

$$\frac{\partial}{\partial x}\left[\frac{1}{E}\left(k^2 E_{xx} + I_n - I_p\right)\right] - \left(I_n + I_p\right) - k^2 E E_x = 0. \tag{11}$$

Assuming that I_n and I_p are functions of t we obtain

$$k^2 E_{xx} - \tfrac{1}{2}k^2 E^3 - \left(I_n + I_p\right) xE + I_n - I_p = 2c_0 E \tag{12}$$

after integrating Eq. (11) with respect to x.

Eq. (12) can also be studied as the equation with control if we take into account the dependences I_n and I_p of t [11]. The assumptions $c_0 = 0$ and

$$E = 2y\left(\frac{k^2}{I_n + I_p}\right)^{-1/3}, \quad x = z\left(\frac{k^2}{I_n + I_p}\right)^{1/3},$$

$$\alpha = \frac{I_p - I_n}{2\left(I_p + I_n\right)} \tag{13}$$

give the second Painlevé equation in the form (1).

The set of equations (2)–(4) can be used to obtain the asymptotic solution of Eq. (12).

Let us look for solutions of Eqs. (2)–(4) in the form

$$p = p_0 + k^2 p_1 + k^4 p_2 + \dots, \tag{14}$$

$$n = n_0 + k^2 n_1 + k^4 n_2 + \dots, \tag{15}$$

$$E = E_0 + k^2 E_1 + k^4 E_2 + \dots \tag{16}$$

Substituting Eqs. (14)–(16) into set of equations (2)–(4) and equating the expressions obtained at different powers of k^2 gives the following set of equations: at k^0:

$$p_0 - n_0 = 0, \tag{17}$$

$$n_{0x} + n_0 E_0 = I_n, \tag{18}$$

$$p_{0x} - p_0 E_0 = I_p, \tag{19}$$

at k^2:

$$p_1 - n_1 = E_{0x}, \tag{20}$$

$$n_{1x} + n_1 E_0 + n_0 E_1 = 0, \tag{21}$$

$$p_{1x} - p_1 E_0 - p_0 E_1 = 0 \tag{22}$$

and at k^4:

$$p_2 - n_2 = E_{1x}, \tag{23}$$

$$n_{2x} + n_2 E_0 + n_1 E_1 + n_0 E_2 = 0, \tag{24}$$

$$p_{2x} - p_2 E_0 - p_1 E_1 - p_0 E_2 = 0. \tag{25}$$

It is easy to find that

$$n_0 = p_0 = \tfrac{1}{2}\left(I_n + I_p\right) x, \tag{26}$$

$$E_0 = \frac{I_n - I_p}{n_0 + p_o} = \frac{2\left(I_n - I_p\right)}{\left(I_n + I_p\right) x} \tag{27}$$

from Eqs. (17)–(19).

Solving Eqs. (20)–(22) one can also obtain

$$E_1 = \frac{E_{0xx}}{n_0 + p_0} - \frac{E_0^3}{2\left(n_0 + p_0\right)} \tag{28}$$

and finally we get

$$E_2 = \frac{E_{1xx}}{n_0 + p_0} + \frac{E_0^2 E_1}{2\left(n_0 + p_0\right)} \tag{29}$$

taking into account Eqs. (23)–(25).

Using Eqs. (27)–(28) one can write down the asymptotic solution of Eq. (1) at $z \to \infty$. It has the form

$$y \simeq -\frac{\alpha}{z} + \frac{2\alpha\left(\alpha^2 - 1\right)}{z^4} + \frac{4\alpha\left(\alpha^2 - 1\right)\left(\alpha^2 + 10\right)}{z^7} + \dots \tag{30}$$

This asymptotic solution is already known [12].

The set of equations (2)–(4) can be presented in the form

$$p_x - 2py = \tfrac{1}{2} + \alpha, \tag{31}$$

$$n_x + 2ny = \tfrac{1}{2} - \alpha, \tag{32}$$

$$y_x = \tfrac{1}{2}\left(p - n\right) \tag{33}$$

if we take the following variables,

$$p = Ap', \quad n = An', \quad A = 2k^2\left(\frac{k^2}{I_n + I_p}\right)^{-2/3},$$

$$x = x'\left(\frac{k^2}{I_n + I_p}\right)^{1/3}, \quad E = 2y\left(\frac{k^2}{I_n + I_p}\right)^{-1/3},$$

$$\frac{I_p - I_n}{2\left(I_n + I_p\right)} = \alpha \tag{34}$$

in the set of equations (2)–(4) and the primes of the variables omit.

The set of equations (31)–(33) is the equivalent to the second Painlevé equation (1). We have

$$p = \frac{y_{xx} - \alpha}{2y} + y_x, \tag{35}$$

$$n = \frac{y_{xx} - \alpha}{2y} - y_x \tag{36}$$

from Eqs. (31) and (32) keeping in mind Eq. (33).

Furthermore we obtain

$$2py = y_{xx} - \alpha + 2yy_x, \tag{37}$$

$$2ny = y_{xx} - \alpha - 2yy_x \tag{38}$$

from Eqs. (35) and (36). Taking into account (37) and (38) we find

$$p = y_x + y^2 + \tfrac{1}{2}x, \tag{39}$$

$$n = -y_x + y^2 + \tfrac{1}{2}x \tag{40}$$

from Eqs. (31) and (32).

Substituting Eqs. (35) and (36) into Eqs. (39) and (40) gives the second Painlevé equation in the form (1).

The set of equations (31), (32), (39) and (40) can be applied to look for special solutions of the second Painlevé equation (1).

Assuming $p = 0$ and $\alpha = -1/2$ in Eq. (31) we get the special solution of Eq. (1) in the form of an Airy function solving Eq. (39) [13,14]. Another special solution of Eq. (1) can be found at $n = 0$ and $\alpha = \frac{1}{2}$ from Eq. (40).

One can also suggest an iterative process to obtain some special solutions taking into account Eqs. (31), (32), (39) and (40).

Let $y(x,\alpha)$ and $y'\left(x,\alpha'\right)$ be solutions of Eq. (1). Then we have

$$p_x - 2py - \tfrac{1}{2} - \alpha = 0, \tag{41}$$

$$y_x + y^2 + \tfrac{1}{2}x - p = 0, \tag{42}$$

and

$$n_x + 2ny' - \tfrac{1}{2} + \alpha' = 0, \tag{43}$$

$$y'_x - y'^2 - \tfrac{1}{2}x + n = 0 \tag{44}$$

from Eqs. (31), (32), (39) and (40).

Now we can obtain the following equations for p and n,

$$\frac{p_{xx}}{p} - \frac{p_x^2}{4p^2} + \frac{(1+2\alpha)^2}{4p^2} + \frac{x}{2} - p = 0, \tag{45}$$

$$\frac{n_{xx}}{2n} - \frac{n_x^2}{4n^2} + \frac{\left(1-2\alpha'\right)^2}{4n^2} + \frac{x}{2} - n = 0 \tag{46}$$

from Eqs. (41), (42) and (43), (44).

We can see that Eqs. (45) and (46) coincide when $p = n$, $\alpha' = -\alpha$ or when $p = n$, $\alpha' = \alpha + 1$. In the former case we have $y'\left(x, \alpha'\right) = -y\left(x, -\alpha'\right)$ from

$$y'\left(x, \alpha'\right) = -y\left(x, \alpha\right) - \frac{\alpha' + \alpha}{2p}. \tag{47}$$

In the latter case we obtain the formula

$$y'\left(x, \alpha'\right) = y\left(x, \alpha + 1\right) = -y\left(x, \alpha\right) - \frac{2\alpha + 1}{2y_x\left(x, \alpha\right) + 2y^2\left(x, \alpha\right) + x} \tag{48}$$

from Eq. (47), which allows us to find the special and rational solutions of Eq. (1). For example, if we take the trivial solution $y_0 = 0$ of Eq. (1) at $\alpha = 0$ we get the solution $y_1 = -1/x$ of Eq. (1) at $\alpha = 1$ and so on.

Eq. (48) was obtained earlier in Refs. [15–17] but we have found this using Eqs. (31)–(33), corresponding to the physical model (2)–(4).

It should be pointed out that the set of equations (2)–(4) to describe the electric field in a semiconductor can be used in the solution of other problems where there are two types of particles. We have found that this set of equations has been applied as a stationary model in the theory of spherical lightning [18], to describe the electric field in a layer near electrodes in a plasma [19] and so on.

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