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The first and second Painlevé equations of higher order and some relations between them

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Abstract

The first and second Painlevé equations of higher order are introduced. The relations between the Korteweg–de Vries hierarchies and their singular manifold equations are presented. These identities are used to search for the relations between the first and the second Painlevé equations of higher order.

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1. Introduction

The second-order ordinary differential equation (ODE) class has been studied by Painlevé and Gambier, who found six new functions which were defined by nonlinear ODEs depending on complex parameters. This important result led to the problem of finding new functions which could be defined by nonlinear ODEs like the Painlevé functions [1]. One of the possible ways in this direction is using the Painlevé equations of higher order.

In this Letter we introduce the first and second Painlevé equations of higher order. Their solutions can be new functions but we do not solve this problem. In fact there are different points of view on this question. For example, Gromak believes that such functions can be new ones, but the proof of this point is very difficult [2]. On the contrary Okamoto thinks these functions do not have to be new ones [3]. Later we only find some relations for the first and second Painlevé equations of higher order which can be useful when proving some properties for these equations. These relations can also be used to seek some exact solutions for the first and second Painlevé equations of higher order.

In order to determine the first and second Painlevé equations of higher order we will apply the following approach.

The current interest in the Painlevé property is known to stem from the observation by Ablowitz and Segur that reductions of a partial differential equation (PDE) of the soliton type give rise to ODEs

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whose movable singularities are only poles. Using this idea some Painlevé equations can be obtained [4–7].

Let us consider two well known examples which illustrate the obtaining of the Painlevé equations. We start with the Korteweg–de Vries equation (KdV)

$$\omega_t + 3\omega\omega_x + \omega_{xxx} = 0 \tag{1.1}$$

and take the following variables [7,8],

$$\omega = F(\theta) - \lambda t, \quad \theta = x + \tfrac{3}{2}\lambda t^2. \tag{1.2}$$

We also have that the equation for $F(\theta)$ when integrated, takes the form of the first Painlevé equation,

$$F_{\theta\theta} + \tfrac{3}{2}F^2 - \lambda\theta + k_1 = 0, \tag{1.3}$$

where λ and k_1 are constants.

As another example we employ the modified Korteweg–de Vries equation (mKdV),

$$u_t - \tfrac{3}{2}u^2 u_x + u_{xxx} = 0, \tag{1.4}$$

and we look for the solution of this equation in the form [7]

$$u = \frac{1}{(3t)^{1/3}} f(\xi), \quad \xi = \frac{x}{(3t)^{1/3}}. \tag{1.5}$$

After integrating we obtain the second Painlevé equation,

$$f_{\xi\xi} - \tfrac{1}{2}f^3 - \xi f + k_2 = 0, \tag{1.6}$$

where k_2 is constant.

These two examples show that the first and second Painlevé equations can be obtained from the KdV and mKdV equations using substitutions (1.2) and (1.5).

2. The first and second Painlevé equations of higher order

In order to introduce the first and second Painlevé equations of higher order we use the KdV and mKdV hierarchies.

In the first place let us take the KdV hierarchy in the form

$$\omega_t + \frac{\partial}{\partial x} B^n(\omega) = 0 \quad (n = 1, 2, \ldots), \tag{2.1}$$

where B^n is determined by the Lenard recursion relation [9]

$$\frac{\partial}{\partial x} B^{n+1}(\omega) = B^n_{xxx} + 2\omega B^n_x + \omega_x B^n \quad (n = 1, 2, \ldots),$$
$$B^0 = 1, \quad B^1 = \omega. \tag{2.2}$$

Regardless of Eqs. (2.2) and (1.2) let us introduce the operator b^n which is determined by the Lenard recursion relation as well,

$$\frac{d}{d\theta} b^{n+1}(F) = b^n_{\theta\theta\theta} + F b^n_\theta + F_\theta b^n \quad (n = 1, 2, \ldots),$$
$$b^0 = 1, \quad b^1 = F, \tag{2.3}$$

where F is a function of θ.

Let us use the singular manifold equations for the KdV hierarchy,

$$Z_t + Z_x B^n(\{z; x\}) = 0, \quad \{z; x\} = \frac{z_{xxx}}{z_x} - \frac{3}{2}\frac{z_{xx}^2}{z_x^2}, \tag{2.4}$$

and look for solutions of this equation in the form

$$Z(x, t) = \Psi(\theta), \quad \theta = [(2n+1)\lambda t]^m,$$
$$m = -\frac{1}{2n+1}. \tag{2.5}$$

Then we obtain the following proposition.

Proposition 2.1. Let expressions (2.2), (2.3) and (2.5) be given. Then there are relations

$$B^n(\{z; x\}) = [(2n+1)\lambda t]^{-2n/(2n+1)} b^n(F) \quad (n = 1, 2, \ldots), \tag{2.6}$$

where

$$F = \{\Psi; \theta\} = \frac{\Psi_{\theta\theta\theta}}{\Psi_\theta} = \frac{3\Psi_{\theta\theta}^2}{\Psi_\theta^2}.$$

Proof. Taking into account formulae (2.2), (2.3) and (2.5) one can obtain equality (2.6) by the mathematical induction method.

Using substitutions (2.5) and equality (2.6) one can obtain the following equation from the singular manifold equations (2.4),

$$b^{n+1}(F) - \lambda\theta = 0 \quad (n = 0, 1, 2, \ldots). \tag{2.7}$$

Let us call Eqs. (2.7) the Painlevé equations of the $2n$th order and denote them by ${}_{2n}P_1$, because we have the first Painlevé equation (1.3) from (2.7) at $n=1$.

If we take $n=2$ we obtain the first Painlevé equation of the fourth order (${}_4P_1$)

$$F_{\theta\theta\theta\theta}+5FF_{\theta\theta}+\tfrac{5}{2}F_\theta^2+\tfrac{5}{2}F^3-\lambda\theta=0.$$

Remark 2.1. Taking into account the conjecture by Ablowitz, Ramani and Segur about the Painlevé property [6,7] we obtain that Eqs. (2.7) have the Painlevé property because these equations were obtained from Eqs. (2.4) which are solvable by the inverse scattering transform.

The solutions of Eqs. (2.7) can be called the Painlevé functions for the Painlevé equations of higher order.

Now let us introduce the second Painlevé equations of higher order using the modified Korteweg–de Vries hierarchy.

We take the mKdV hierarchy in the form

$$u_t+\frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial x}+u\right)B^n\left(u_x-\tfrac{1}{2}u^2\right)\right]=0$$
$$(n=0,1,2,\ldots), \qquad (2.8)$$

where the operator B^n is also determined by formula (2.2). However, operator B^n acts on expression $u_x-\frac{1}{2}u^2$ in this case.

Let us introduce the operator d^n which is also determined by formula (2.3). Let us seek solutions of Eqs. (2.8) in the form

$$u=[(2n+1)t]^m f(\xi),\quad \xi=x[(2n+1)t]^m,$$
$$m=-\frac{1}{2n+1}. \qquad (2.9)$$

Then taking into account expressions (2.2), (2.3) and variables (2.9) one can formulate the following proposition.

Proposition 2.2. Let (2.2), (2.3) and (2.9) be given, then we have the equality

$$B^n\left(u_x-\tfrac{1}{2}u^2\right)$$
$$=[(2n+1)t]^{-2n/(2n+1)}d^n\left(f_\xi-\tfrac{1}{2}f^2\right)$$
$$(n=1,2,\ldots). \qquad (2.10)$$

Proof. We apply the mathematical induction method for the proof. If we take $n=1$ we then have the equality. Assuming the validity of relation (2.10) at $n=k$ we obtain the relation at $n=k+1$ using expressions (2.2), (2.3) and variables (2.9).

Substitution of expressions (2.9) and (2.10) into Eq. (2.8) leads to the following equation,

$$\left(\frac{\mathrm{d}}{\mathrm{d}\xi}+f\right)d^n\left(f_\xi-\tfrac{1}{2}f^2\right)-\xi f+k_2=0$$
$$(n=1,2,\ldots), \qquad (2.11)$$

after integration of ξ.

We have the second Painlevé equation (1.3) from Eq. (2.11) at $n=1$ and therefore we call Eqs. (2.11) the second Painlevé equations of the $2n$th order and we denote them by ${}_{2n}P_2$. One can call the solution of these equations the Painlevé functions of the second Painlevé equations of $2n$th order.

If we take $n=2$ in Eq. (2.11) we obtain the second Painlevé equation of fourth order (${}_4P_2$)

$$f_{\xi\xi\xi\xi}-\tfrac{5}{2}f^2f_{\xi\xi}-\tfrac{5}{2}ff_\xi^2+\tfrac{3}{8}f^5-\xi f+k_2=0. \qquad (2.12)$$

Remark 2.2. Eqs. (2.11) have the Painlevé property like Eqs. (2.7). These equations were obtained by reductions of the equations of the mKdV hierarchy (2.8) which are also integrable by the inverse scattering transform.

Eqs. (2.7) and (2.11) are equations of the first and second Painlevé equation hierarchies. We have to note that Eq. (2.11) was considered also in Ref. [10], where it was obtained by another reduction.

3. Sufficiency condition for the integrability of some nonlinear PDEs

In order to obtain the relations between the first and second Painlevé equations of higher order let us use the relations which were obtained for some nonlinear PDEs in Refs. [11–13].

A particular result of the Painlevé test when applied to PDEs is known to be the singular manifold equation that corresponds to the original equation [14,15]. However, a number of classes of nonlinear PDEs turned out to have identities which connect the original equation to their singular manifold equation.

To be more exact let us demonstrate such relations for two classes of nonlinear PDEs.

In the first place let us consider the family of equations which take the form

$$u_t + \frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial x} + u\right) K\left(u_x - \tfrac{1}{2}u^2\right)\right] = 0, \qquad (3.1)$$

where K are operators or functions of $u_x - \frac{1}{2}u^2$.

We also take the family of equations in the form

$$z_t + z_x K(\{z;\, x\}) = 0, \qquad (3.2)$$

where $\{z;\, x\}$ is the Schwarzian derivative

$$\{z;\, x\} = \frac{z_{xxx}}{z_x} - \frac{3}{2}\frac{z_{xx}^2}{z_x^2}. \qquad (3.3)$$

If we search for solutions of Eqs. (3.1) in the form of truncated expansions [16,17]

$$u = -\frac{2 z_x}{z} + \frac{z_{xx}}{z_x}, \qquad (3.4)$$

then we can obtain the following proposition.

Proposition 3.1. Let solutions of Eqs. (3.1) be determined by formula (3.4), then the following identity is correct,

$$u_t + \frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial x} + u\right) K\left(u_x - \tfrac{1}{2}u^2\right)\right] = \frac{\partial}{\partial x}\left(\frac{1}{z_x}\frac{\partial}{\partial x} - \frac{2}{z}\right)\left[z_t + z_x K(\{z;\, x\})\right]. \qquad (3.5)$$

Proof. If we take into account the equality

$$u_x - \tfrac{1}{2}u^2 = \{z;\, x\}, \qquad (3.6)$$

then substitutions (3.4) into Eqs. (3.1) lead to identity (3.5).

Remark 3.1. If we look for solutions of Eq. (3.1) in the form

$$u = \frac{z_{xx}}{z_x},$$

then we can obtain for Eq. (3.1) another equality

$$u_t + \frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial x} + u\right) K\left(u_x - \tfrac{1}{2}u^2\right)\right] = \frac{\partial}{\partial x}\left(\frac{1}{z_x}\frac{\partial}{\partial x}\left[z_t + z_x K(\{z;\, x\})\right]\right). \qquad (3.7)$$

In the second place let us consider the class of equations

$$\omega_t + K_{xxx} + \omega_x K + 2\omega K_x = 0,$$
$$K = K(\omega,\, \omega_x, \ldots). \qquad (3.8)$$

Let us assume that the solution of Eqs. (3.8) is determined by the formula

$$\omega = \{z;\, x\}, \qquad (3.9)$$

where $z = z(x, t)$ is the solution of Eqs. (3.2) too, then we meet the proposition.

Proposition 3.2. Let ω be determined by formula (3.9), then we have the identity

$$\omega_t + K_{xxx} + \omega_x K + 2\omega K_x = \left(\frac{z_{xx}}{z_x} - \frac{2 z_x}{z} - \frac{\partial}{\partial x}\right)\frac{\partial}{\partial x}\left(\frac{2}{z} - \frac{1}{z_x}\frac{\partial}{\partial x}\right) \times \left[z_t + z_x K(\{z;\, x\})\right]. \qquad (3.10)$$

Proof. Taking into account the Miura transformation

$$\omega = u_x - \tfrac{1}{2}u^2 \qquad (3.11)$$

and equality (3.5) we obtain identity (3.10).

Using relations (3.5) and (3.10) one can introduce the following sufficiency condition for integrability of the families of equations (3.1) and (3.8).

Proposition 3.3. Let Eqs. (3.2) be integrable, then Eqs. (3.1) and (3.8) are integrable as well.

Proof. If the initial condition for Eqs. (3.1) is given, then one can obtain the initial condition for Eqs. (3.2), which are integrable due to the proposition. Solutions of Eqs. (3.1) are determined by formula (3.4) for known solutions of Eqs. (3.2).

Weiss considered Eqs. (3.2) where $K(\{z;\, x\})$ had the polynomial form in $\{z;\, x\}$ and obtained a number of classes of integrable equations of type (3.2) [9]. Using this idea we suggested the integrability test for Eqs. (3.1) and (3.8) [18]. It is important to note that this test gives all integrable equations of polynomial form which are presented in Ref. [19]. In the general case Eqs. (3.1) and (3.8) are not integrable, because Eqs. (3.2) are not integrable as well.

For the integrable equation (3.1) and (3.8) we have that Eqs. (3.2) are the singular manifold equations for these equations.

For example if we take

$$K\left(u_x-\tfrac{1}{2}u^2\right)=B^n\left(u_x-\tfrac{1}{2}u^2\right) \tag{3.12}$$

we obtain the mKdV hierarchy (2.8) in this case.

Taking into account propositions (3.1) and (3.2) we obtain the following relations for Eqs. (2.1) and (2.8),

$$u_t+\frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial x}+u\right)B^n\left(u_x-\tfrac{1}{2}u^2\right)\right] =\frac{\partial}{\partial x}\left(\frac{1}{z_x}\frac{\partial}{\partial x}-\frac{2}{z}\right)\left[z_t+z_xB^n(\{z;\,x\})\right], \tag{3.13}$$

$$\omega_t+\frac{\partial}{\partial x}B^{n+1}(\omega) =\left(\frac{z_{xx}}{z_x}-\frac{2z_x}{z}-\frac{\partial}{\partial x}\right)\frac{\partial}{\partial x}\left(\frac{2}{z}-\frac{1}{z_x}\frac{\partial}{\partial x}\right) \times\left[z_t+z_xB^n(\{z;\,x\})\right]. \tag{3.14}$$

Let us note that Eqs. (2.4) are the singular manifold equations for Eqs. (2.1) and (2.8) [9].

We are going to use identities (3.13) and (3.14) to obtain relations between the first and the second Painlevé equations of higher order.

4. Relations between the first and second Painlevé equations of higher order

Let us look for the solution of Eq. (2.4) using the following variables,

$$z=z(\xi),\quad \xi=x[(2n+1)t]^{-1/(2n+1)}, \tag{4.1}$$

then, taking into account Eq. (2.10), we have

$$d^n(\{z;\,\xi\})-\xi=0. \tag{4.2}$$

Using variables (2.9) and (4.1) we obtain the following relations,

$$\left(\frac{\mathrm{d}}{\mathrm{d}\xi}+f\right)d^n\left(f_\xi-\tfrac{1}{2}f^2\right)-\xi f-1 =\left(\frac{1}{z_\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}z_\xi-\frac{2z_\xi}{z}\right)\left[d^n(\{z;\,\xi\})-\xi\right], \tag{4.3}$$

$(n=1,2,\ldots)$

from identity (3.13).

We can see from relation (4.3) that the left-hand side of Eq. (4.3) contains the second Painlevé equations of the $2n$th order whereas the right-hand side represents the first Painlevé equation of $(2n-2)$th order using the notation

$$F=f_\xi-\tfrac{1}{2}f^2=\{z;\,\xi\}. \tag{4.4}$$

Relations (4.3) can be transformed to the following ones,

$$\left(\frac{\mathrm{d}}{\mathrm{d}\xi}+f\right)d^n\left(f_\xi-\tfrac{1}{2}f^2\right)-\xi f-1 =\left(\frac{\mathrm{d}}{\mathrm{d}\xi}+f\right)\left[d^n(F)-\xi\right], \tag{4.5}$$

if we take into account that

$$f=-\frac{2z_\xi}{z}+\frac{z_{\xi\xi}}{z_\xi}. \tag{4.6}$$

We can formulate the following proposition.

Proposition 4.1. Let F be a solution of the first Painlevé equations of the $(2n-2)$th order,

$$d^n(F)-\xi=0\quad (n=1,2,\ldots), \tag{4.7}$$

then the solution $f(\xi)$ of the second Painlevé equation of the $2n$th order (2.11) at $k_2=-1$ can be obtained as the solution of the equation

$$f_\xi-\tfrac{1}{2}f^2=F. \tag{4.8}$$

Proof. This follows from equality (4.5) because this relation gives the connection between solutions of equations ${}_{2n}P_1$ and ${}_{(2n-2)}P_2$.

Taking into account the formula

$$f=\frac{2\Psi_\xi}{\Psi} \tag{4.9}$$

one can obtain the linear equation

$$\Psi_{\xi\xi}+\tfrac{1}{2}F\Psi=0 \tag{4.10}$$

from Eq. (4.8).

Eq. (4.10) shows that if we had known the solution F of the first Painlevé equation of higher order (2.7) we could have found the solution f of the second Painlevé equation (2.11) using Eqs. (4.9).

Let us note that the assumption $n=1$ in relation (4.5) gives the equality

$$f_{\xi\xi}-\tfrac{1}{2}f^3-\xi f-1=\left(\frac{d}{d\xi}+f\right)(F-\xi) \quad (4.11)$$

and we obtain the well-known result that the solution of the second Painlevé equation is determined by Eq. (4.10) where $F=\xi$ [20].

We do not have any exact solutions of Eqs. (4.2) at $n>1$ but we know the asymptotic solutions of this equation at $n=2$ [21] which can be used to look for the asymptotic solutions of Eq. (2.12).

Using the self-similar variables

$$\omega=\left[(2n+1)t\right]^{2m}F(\xi),\quad \xi=x\left[(2n+1)t\right]^{m},$$
$$m=-\frac{1}{2n+1}, \quad (4.12)$$

one can obtain the relations from identity (3.14) which connect the first Painlevé equation of higher order (2.7) and the ODEs

$$\frac{d}{d\xi}d^{n+1}(F)-\xi F_\xi-2F=0. \quad (4.13)$$

This relation is of the form

$$\left[\frac{d}{d\xi}d^{n+1}(F)-\xi F_\xi-2F\right]$$
$$=\left(\frac{d}{d\xi}-f\right)\frac{d}{d\xi}\left(\frac{d}{d\xi}+f\right)[d^n)(F)-\xi]$$
$$(n=1,2,\dots), \quad (4.14)$$

where

$$F=f_\xi-\tfrac{1}{2}f^2. \quad (4.15)$$

Relation (4.14) shows that Eqs. (4.13) have some solutions which are also solutions of the first Painlevé equation of higher order.

We can see from identity (4.14) that the equation of third order

$$F_{\xi\xi\xi}+3FF_\xi-2F-\xi F_\xi=0, \quad (4.16)$$

which is obtained at $n=1$ from Eq. (4.13) has a rational solution $F=\xi$ and this solution leads to the evident solution of the KdV equation (1.1) in the form

$$\omega=\frac{x}{3t}. \quad (4.17)$$

Comparison of relations (4.14) and (4.5) shows that Eqs. (4.13) have the connection with the second Painlevé equation of higher order in the form

$$\frac{d}{d\xi}d^{n+1}(F)-\xi F_\xi-2F$$
$$=\left(\frac{d}{d\xi}-f\right)\frac{d}{d\xi}\left[\left(\frac{d}{d\xi}+f\right)d^n\left(f_\xi-\tfrac{1}{2}f^2\right)\right.$$
$$\left.-\xi f-1\right]. \quad (4.18)$$

This relation allows one to find solutions of Eqs. (4.13) using the solutions of the second Painlevé equation of higher order (2.10) at $k_2=-1$ and formula (4.15).

5. Generalization for the second Painlevé equation in two dimensions

Let us show that one can obtain the generalization for the second Painlevé equation in two dimensions taking into account the modified Kadomtsev–Petviashvili (mKP) equation.

The mKP equation is of the form [21,22]

$$v_{xt}-\tfrac{1}{4}v_{xxxx}+\tfrac{3}{2}v_x^2v_{xx}+\tfrac{3}{2}v_{xx}v_y-\tfrac{3}{4}v_{yy}=0. \quad (5.1)$$

Let us look for the solution of Eq. (5.1) in the form

$$v=g(\xi,\theta),\quad \xi=\frac{x}{(3t)^{1/3}},\quad \theta=\frac{y}{(3t)^{2/3}}, \quad (5.2)$$

then substitution of (5.2) into Eq. (5.1) leads to the following equation,

$$\frac{\partial}{\partial\xi}\left(\tfrac{1}{4}g_{\xi\xi\xi}-\tfrac{1}{2}g_\xi^3+\xi g_\xi+2\theta g_\theta\right)$$
$$+\tfrac{3}{4}g_{\theta\theta}-\tfrac{3}{2}g_{\xi\xi}g_\theta=0. \quad (5.3)$$

This equation can be transformed if we assume $g_\xi=f$. It takes the form

$$\tfrac{1}{4}f_{\xi\xi}-\tfrac{1}{2}f^3+\xi f+2\theta\partial_\xi^{-1}f_\theta$$
$$+\tfrac{3}{4}\partial_\xi^{-1}\left(\partial_\xi^{-1}f_{\theta\theta}-2f_\xi\partial^{-1}f_\theta\right)=0,$$
$$\partial_\xi^{-1}f=\int_{-\infty}^{\xi}f\,d\xi, \quad (5.4)$$

after integration of ξ.

It is obvious that Eq. (5.4) at $f_\theta = 0$ gives the second Painlevé equation (1.6). Therefore, one can call Eq. (5.4) the second Painlevé equation in two dimensions.

We expect that Eq. (5.3) can be integrated by the inverse scattering transform.

It was shown in Ref. [12] that the mKP equation could be presented as the identity connecting the original equation with its singular manifold equation (SME).

The SME for the KP and mKP equations takes the form [24]

$$z_t - \tfrac{1}{4} z_x\{z;\, x\} - \frac{3}{8}\frac{z_y^2}{z_x} - \tfrac{3}{4} z_x \partial_x^{-1}\left(\frac{z_y}{z_x}\right)_y = 0, \tag{5.5}$$

where

$$\partial_x^{-1} u = \int_0^x u \,\mathrm{d}\xi. \tag{5.6}$$

If we take $u = v_x$ in Eq. (5.1) and assume

$$u = -\frac{z_x}{z} + \frac{z_{xx}}{2 z_x} - \frac{z_y}{z_x} \tag{5.7}$$

we obtain the equality [12]

$$u_t - \tfrac{1}{4} u_{xxx} + \tfrac{3}{2} u^2 u_x + \tfrac{3}{2} u_x \partial_x^{-1} u_y - \tfrac{3}{4} \partial_x^{-1} u_{yy}$$
$$= \left[\frac{\partial}{\partial x}\left(\frac{1}{2 z_x}\frac{\partial}{\partial x} - \frac{1}{z}\right) - \frac{1}{2 z_x}\frac{\partial}{\partial y} + \frac{z_y}{2 z_x^2}\frac{\partial}{\partial x}\right]$$
$$\times \left[z_t - \tfrac{1}{4} z_x\{z;\, x\} - \frac{3}{8}\frac{z_y^2}{z_x} - \tfrac{3}{4} z_x \partial_x^{-1}\left(\frac{z_y}{z_x}\right)_y\right]. \tag{5.8}$$

Eq. (5.5) has the solution in the form

$$z = P(\xi, \theta), \quad \xi = \frac{x}{(3t)^{1/3}}, \quad \theta = \frac{y}{(3t)^{2/3}}, \tag{5.9}$$

where $P(\xi, \theta)$ has to satisfy to the following equation,

$$\{P;\, \xi\} + \frac{3}{2}\frac{P_\theta^2}{P_\xi^2} + 3\partial_\xi^{-1}\left(\frac{P_\theta}{P_\xi}\right)_\theta + 4\xi + 8\theta\frac{P_\theta}{P_\xi} = 0, \tag{5.10}$$

which is obtained from Eq. (5.5). This equation is the two-dimensional generalization of Eq. (4.12) in two dimensions.

If we take into account self-similar variables (5.2) and (5.9), then equality (5.8) transforms to the identity which connect Eqs. (5.4) and (5.10). The solutions of Eq. (5.4) can be found with the formula

$$f(\xi, \theta) = -\frac{P_\xi}{P} + \frac{P_{\xi\xi}}{2 P_\xi} - \frac{P_\theta}{P_\xi} \tag{5.11}$$

for known solutions of Eq. (5.10).

One can also offer the generalization of the second Painlevé equation of higher order in two dimensions if we take into account the modified Kadomtsev–Petviashvili hierarchy.

6. Conclusion

Let us emphasize the results of this work. Using the reductions from the KdV and the mKdV hierarchies to ODEs we introduce the first and second Painlevé equations of higher order. The equalities connecting the KdV and mKdV hierarchies with their singular manifold equations were presented. These identities were used to find the relations between the first and second Painlevé equations of higher order. The generalization of the second Painlevé equation in two dimensions was offered using the modified Kadomtsev–Petviashvili equation.

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