J. Phys. A: Math. Gen. 27 (1994) 2457–2470. Printed in the UK

From singular manifold equations to integrable evolution equations

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Received 3 August 1993, in final form 5 November 1993

Abstract. The singular manifold equations are used to construct nonlinear integrable partial differential equations. A number of well known integrable equations (hierarchies of Korteveg–de Vries, Caudrey–Dodd–Gibbon, Kaup–Kuperschmidt, and Harry Dym) are obtained. Some new integrable equations are presented. A possible approach to classification of integrable evolution equations is discussed.

1. Introduction

In this paper we use the equations

$$z_t + z_x G(\{z;x\}) = 0 \tag{1.1}$$

where $G(\omega)$ are operators or functions of

$$\omega = \{z;x\} = \frac{z_{xxx}}{z_x} - \frac{3}{2}\frac{z_{xx}^2}{z_x^2} \tag{1.2}$$

to construct integrable evolution equations.

Equations (1.1) often arise when nonlinear partial differential equations are studied for the Painleve property [1–7].

The advantage of (1.1) is that its solutions are invariant under the action Möbius group on z [5, 8, 9]

$$z \to \frac{az+b}{cz+d} \qquad ab - cd = 1\,. \tag{1.3}$$

Equations (1.1) can be written as functions of two elementary homographic invariants [8, 9]

$$C = G(S) \tag{1.4}$$

where

$$C = -\frac{z_t}{z_x} \qquad S = \{z;x\}\,.$$

We can see from (1.4) that (1.1) are the simplest ones which are invariant under the Möbius group.

Later in this paper we show that (1.1) give a reasonable grouping of the most common integrable equations and allow us to look for new integrable partial differential equations.

2. The singular manifold equations

We will call (1.1) singular manifold equations.

Some equations of family (1.1) are well known as integrable partial differential equations.

In fact we have for

$$G=\omega=\{z;x\} \tag{2.1}$$

the Krichever–Novikov equation [10]

$$z_t+z_{xxx}-\frac{3z_{xx}^2}{2z_x}=0. \tag{2.2}$$

The equations of family (1.1) were studied in [11] when $G(\omega)$ was a function of ω. In [11] it was obtained that (1.1) are integrable equations in the cases

$$G(\omega)=C_1+C_2\omega \tag{2.3}$$

$$G(\omega)=C_1-(C_2+C_3\omega)^{-2} \tag{2.4}$$

$$G(\omega)=C_1-\frac{C_3+2C_4\omega}{\sqrt{C_2+C_3\omega+C_4\omega^2}} \tag{2.5}$$

where C_1, C_2, C_3 and C_4 are arbitrary constants.

Weiss has studied the equations of family (1.1) in detail and has shown that these equations will possess identically the Painleve property, when there exists a transformation [12]

$$z_x=\psi_x^m \tag{2.6}$$

where m is rational and negative and ψ satisfies (1.1).

In particular, Weiss has found that the singular manifold equations

$$z_t+z_xb^{n+1}(\{z;x\})=0 \tag{k_s}$$

where b^n is determined by the Lenard recursion relation [12]

$$\frac{\partial}{\partial x}b^{n+1}=b^n_{xxx}+2\omega b^n_x+\omega_x b^n \qquad (n=1,2,\ldots)$$
$$b^0=1 \qquad b^1=\omega \tag{2.7}$$

have transformation (2.6) at $m=-1$ and give the Korteveg–de Vries (KdV) hierarchy [12].

The singular manifold equations for sequences of Caudrey–Dodd–Gibbon and Kaup–Kuperschmidt equations have the form [12]

$$z_t+z_xH_n(\{z;x\})=0 \tag{h_s}$$

$$z_t+z_xG_n(\{z;x\})=0 \tag{g_s}$$

(operator H_n and G_n are given in [12–17].)

It is known that equations (h_s) and (g_s) possess the symmetries $z_x=\varphi_x^{-2}$ and $z_x=\varphi_x^{-1/2}$ [12].

The singular manifold equations with formulae (2.3)–(2.5), (k_s), (h_s) and (g_s) will be used later to construct nonlinear integrable partial differential equations.

3. The families of modified and Lax equations

We have recently considered the following family of equations [18]:

$$u_t+\frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial x}+u\right)G\left(u_x-\frac{u^2}{2}\right)\right]=0 \tag{3.1}$$

where $G(u_x-u^2/2)$ is an operator or smooth function of $u_x-u^2/2$. Obviously, the family of equations (3.1) contains the modified KdV equation at $G=u_x-u^2/2$.

Let us take two transformations,

$$u=-\frac{2z_x}{z_0+z}+\frac{z_{xx}}{z_x} \tag{3.2}$$

and

$$v=\frac{z_{xx}}{z_x}. \tag{3.3}$$

We then have the following theorems.

Theorem 3.1. If transformations (3.2) and (3.3) are valid, then

$$\omega=u_x-u^2/2=v_x-v^2/2=\{z;x\}. \tag{3.4}$$

Proof. Substituting transformations (3.2) and (3.3) into (3.4) gives equality.

Theorem 3.2. Let $x(z, t)$ be solutions of (1.1), then $u(x, t)$ and $v(x, t)$ by (3.2) and (3.3) give solutions of (3.1).

Proof. It follows from the equations that

$$u_t+\frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial x}+u\right)G\left(u_x-\frac{u^2}{2}\right)\right]=-\frac{\partial}{\partial x}\left[\left(\frac{2}{z_0+z}+\frac{1}{z_x}\frac{\partial}{\partial x}\right)(z_t+z_xG(\{z,x\}))\right] \tag{3.5}$$

$$v_t+\frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial x}+v\right)G\left(v_x-\frac{v^2}{2}\right)\right]=\frac{\partial}{\partial x}\left[\frac{1}{z_x}\frac{\partial}{\partial x}(z_t+z_xG(\{z,x\}))\right]. \tag{3.6}$$

Now let us take the family of Lax equations [19, 20]

$$\omega_t+G_{xxx}+\omega_xG+2\omega G_x=0 \qquad G=G(\omega,\omega_x,\ldots) \tag{3.7}$$

where G in (3.7) are smooth functions or operators of ω.

Theorem 3.3. Let $z(x, t)$ be solutions of (1.1), then

$$\omega = u_x - \frac{u^2}{2} = \{z; x\} \tag{3.8}$$

give solutions of (3.7).

Proof. Taking into account (3.5) and the Miura transformations (3.8) one can get the equality

$$\omega_t + G_{xxx} + \omega_x G + 2\omega G_x = \left(\frac{2z_x}{z_0+z} - \frac{\partial}{\partial x} - \frac{z_{xx}}{z_x}\right)\frac{\partial}{\partial x}\left(\frac{2}{z_0+z} - \frac{1}{z_x}\frac{\partial}{\partial x}\right)[z_t + z_x G(\{z; x\})] \tag{3.9}$$

which proves the theorem.

At first glance it would seem that all equations of families (3.1) and (3.7) have the Painleve property. In fact, we will have three arbitrary functions in the expansion of the solution if $G = u_x - u^2/2$ is taken, as follows from (3.5). Higher-order equations of family (3.1) can have derivatives of $u_x - u^2/2$ with respect to x or t and give a full complement of arbitrary functions because of invariant (3.4). But the Painleve property requires that all movable singularities are single valued in the neighbourhood of a singular manifold $z(x, t) = 0$ [2, 12]. This is not the case for a number of equations of families (3.1) and (3.7). For example, if we take

$$G = \frac{\partial}{\partial x}\left[u_x - \frac{u^2}{2}\right] \tag{3.10}$$

we will get an equation from (3.7) that does not have the Painleve property [2, 12].

Note that similar examples of equations were considered by Clarkson [21]. His second-order equation also has two arbitrary functions, but does not have the Painleve property.

There are equations that both have and do not have the Painleve property among equations of the families (3.1) and (3.7), depending on the form of (1.1).

4. Special solutions of equations (3.1)

It is convenient to use the singular manifold equations (1.1) for obtaining special solutions of (3.1).

Let us consider the solitary wave solution of these equations:

$$Z(x, t) = Z(\xi) \qquad \xi = x - C_0 t. \tag{4.1}$$

We then get

$$G(\{Z; \xi\}) = C_0 \tag{4.2}$$

from (1.1).

Assuming that

$$\{Z; \xi\} = -k^2 \tag{4.3}$$

where k is an arbitrary constant, we have

$$C_0 = G(-k^2). \tag{4.4}$$

Solving (4.3) we find

$$\frac{Z_{\xi\xi}}{Z_\xi} = -2k \tanh(k\xi + \varphi_0) \tag{4.5}$$

$$Z(\xi) = b - a \tanh(k\xi + \varphi_0) \tag{4.6}$$

where φ_0, a and b are arbitrary constants.

Substituting (4.5) and (4.6) into (3.2) and (3.3) gives solutions of (3.1) in the form

$$v = -k \tanh(k\xi + \varphi_0)$$

$$u = \frac{ak \tanh(k\xi + \varphi_0)}{a \tanh(k\xi + \varphi_0) - b} - k \tanh(k\xi + \varphi_0). \tag{4.7}$$

In the case of the concrete function $G(\omega)$ one can find other solutions of (3.1). For example, if we take for $G(\omega)$ (3.11) we will then get the solution of (3.1) in terms of the standard Airy function.

Note that using this approach one can obtain the solution of equations that are more general than (3.1).

Let us consider the following equations:

$$u_t + \frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial x} + u\right) G\left(u_x - \frac{u^2}{2}\right)\right] + Q\left(u_x - \frac{u^2}{2}\right) = 0 \tag{4.8}$$

where Q is smooth function or operator of $u_x - u^2/2$.

Solutions of these equations can be found as solutions of the overdetermined set of equations

$$Z_t + Z_x G(\{Z; x\}) = 0 \tag{4.9}$$

$$Q(\{Z; x\}) = 0. \tag{4.10}$$

For example, let us find a solution of the generalized Burgers–KdV equation

$$u_t + \alpha u u_x - 6u^2 u_x + u_{xxx} = \alpha u_{xx} \tag{4.11}$$

where α is a smooth arbitrary function of x and t.

This equation is obtained from (4.8) at

$$G = u_x - \frac{u^2}{2}$$

and

$$Q = -\alpha \frac{\partial}{\partial x}\left(u_x - \frac{u^2}{2}\right).$$

The overdetermined set of equations (4.9) and (4.10) has the form in this case of

$$Z_t + Z_{xxx} - \frac{3Z_{xx}^2}{2Z_x} = 0 \tag{4.12}$$

$$\alpha(x, t)\frac{\partial}{\partial x}\{Z; x\} = 0. \tag{4.13}$$

Solutions of this set of equations are expressed in terms of (4.7).

For a second example let us take the equation

$$u_t + \frac{1}{2}\frac{\partial}{\partial x}[(u^2)_x - u^3] = 0. \tag{4.14}$$

This equation is frequently referred to as the porous media equation [22–24]. It coincides with (4.8) if we assume

$$G = u_x - \frac{u^2}{2}$$

and

$$Q = -\frac{\partial^2}{\partial x^2}\left(u_x - \frac{u^2}{2}\right).$$

The overdetermined set of equations (4.9) and (4.10) has the form in this case of

$$Z_t + Z_{xxx} - \frac{3Z_{xx}^2}{2Z_x} = 0 \tag{4.15}$$

$$\frac{\partial^2}{\partial x^2}\{Z; x\} = 0. \tag{4.16}$$

Let us look for solutions of set (4.15) and (4.16) as the self-similar solutions

$$Z(x, t) = Z(\vartheta) \qquad \vartheta = \frac{x}{t^{1/3}}. \tag{4.17}$$

We then get from (4.15) and (4.16)

$$\{Z; \vartheta\} = \frac{\vartheta}{3}. \tag{4.18}$$

The solution of this equation takes the form

$$Z(\vartheta) = C_1 \int_0^{\vartheta} \Psi(\xi)^{-2}\, d\xi \tag{4.19}$$

where

$$\Psi(\vartheta) = C_2 \operatorname{Ai}\left(\frac{\vartheta}{3}\right) + C_3 \operatorname{Bi}\left(\frac{\vartheta}{3}\right) \tag{4.20}$$

C_1, C_2, C_3 are arbitrary constants, and $\operatorname{Ai}(y)$ and $\operatorname{Bi}(y)$ are the Airy functions.

The solution of (4.14) can be presented in the form

$$u=\frac{C_1}{t^{1/3}}\left\{\Psi^{-2}\left[C_4-\int_0^{\vartheta}\Psi(\xi)^{-2}\,\mathrm{d}\xi\right]\right\}^{-1}-\frac{1}{t^{1/3}}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\ln\Psi \tag{4.21}$$

where C_1 and C_4 are arbitrary constants.

5. Integrable equations of families (3.1) and (3.7)

Let us present a few integrable equations of families (3.1) and (3.7).

Obviously, (3.1) and (3.7) are integrable equations when (1.1) are also integrable or have the Painleve property.

Substituting (2.3) into (3.1) gives the modified KdV equation, but we can get the new integrable partial differential equations from (3.1) taking into account (2.4) and (2.5).

In particular, if $C_1=C_2=0$ and $C_3=1$ in (2.4) one can obtain the following integrable equation from (3.1):

$$u_t=\frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial x}+u\right)\left(u_x-\frac{u^2}{2}\right)^{-2}\right]=0. \tag{5.1}$$

Assuming $C_1=C_2=C_4=0$ and $C_3=1$ in (2.5), the following integrable equation is obtained:

$$u_t=\frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial x}+u\right)\left(u_x-\frac{u^2}{2}\right)^{-1/2}\right]. \tag{5.2}$$

Substituting equations (k_s), (h_s) and (g_s) into (3.1) gives the following hierarchies of integrable equations:

$$u_t+\frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial x}+u\right)b^{n+1}\left(u_x-\frac{u^2}{2}\right)\right]=0 \qquad (k_m)$$

$$u_t+\frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial x}+u\right)H_n\left(u_x-\frac{u^2}{2}\right)\right]=0 \qquad (h_m)$$

$$u_t+\frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial x}+u\right)G_n\left(u_x-\frac{u^2}{2}\right)\right]=0. \qquad (h_m)$$

The property of possessing a transformation within class (1.1) is additive for (2.6) with the same value of m [12].

For example, the equation

$$z_t+z_x\sum_{n=0}^{N}\alpha_n b^{n+1}(\{z;x\})=0 \tag{5.3}$$

has the auto-Bäcklund transformation

$$z_x=\psi_x^{-1} \tag{5.4}$$

where α_n is a function of t.

We have the following integrable equations from (3.1):

$$u_t+\frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial x}+u\right)\sum_{n=0}^{N}\alpha_n b^{n+1}\left(u_x-\frac{u^2}{2}\right)\right]=0. \tag{5.5}$$

One can obtain rational solutions of (5.5) using the iterative formula [12]

$$\frac{\partial z_{n+1}}{\partial x}=z_n^2\left(\frac{\partial z_n}{\partial x}\right)^{-1}. \tag{5.6}$$

It is clear that equations of family (3.7) will be integrable too, when (1.1) possess the Painleve property or (1.1) are integrable equations.

Substituting (2.3) into (3.7) gives the KdV equation.

We can get new nonlinear integrable partial differential equations from (3.7) taking into account (2.4) and (2.5).

In particular, if $C_1=C_2=0$, $C_3=1$ in (2.4) one can obtain the following integrable partial differential equations from (3.7):

$$\omega_t=(\omega^{-2})_{xxx}-b\omega^{-2}\omega_x. \tag{5.7}$$

Assuming $C_1=C_2=C_4=0$ and $C_3=1$ in (2.5), the following integrable equation is obtained:

$$\omega_t=(\omega^{-1/2})_{xxx}. \tag{5.8}$$

Substituting equations (k_s), (h_s) and (g_s) into (3.7) gives the hierarchies of the KdV, Caudrey–Dodd–Gibbon and Kaup–Kupershmidt equations, respectively [12]:

$$\omega_t+b^{n+2}(\omega)=0 \tag{k_l}$$

$$\omega_t+LH_n(\omega)=0 \tag{h_l}$$

$$\omega_t+LG_n(\omega)=0 \tag{g_l}$$

where the operator L is determined as

$$LF=F_{xxx}+\omega_x F+2\omega F_x.$$

Substituting (5.3) into (3.7) gives the following family of integrable equations:

$$\omega_t+\sum_{n=0}^{N}\alpha_n b^{n+2}(\omega)=0. \tag{5.9}$$

One can also obtain rational solutions of equations (k_l) and (5.9) using (5.6).

6. New hierarchies of integrable equations

We are now interested in studying the family of equations

$$u_t+\frac{\partial}{\partial x}\left[uG\left(\frac{u_{xx}}{u}-\frac{u^2}{2}-\frac{3u_x^2}{2u^2}\right)\right]=0 \tag{6.1}$$

where $G(\omega)$ are smooth functions or operators of

$$\omega=\frac{u_{xx}}{u}-\frac{u^2}{2}-\frac{3u_x^2}{2u^2}. \tag{6.2}$$

Let us seek solutions of (6.1) in the form [25, 26]

$$u=-zw_x \qquad w=\frac{1}{z_0+z}. \tag{6.3}$$

One can obtain [27]

$$-z_{xx}w=u_x-u^2 \tag{6.4}$$

$$-z_{xxx}w=u_{xx}-3uu_x+u^3 \tag{6.5}$$

using (6.3).

We also have

$$u_t=-\frac{\partial}{\partial x}(z_t w). \tag{6.6}$$

One can get the equality

$$u_t+\frac{\partial}{\partial x}\left[2uG\left(\frac{u_{xx}}{u}-\frac{u^2}{2}-\frac{3u_x^2}{2u^2}\right)\right]=-\frac{\partial}{\partial x}[w(z_t+z_xG(\{z;x\}))] \tag{6.7}$$

taking into account (6.3)–(6.6).

We can see from the last expression that solutions of (6.1) are determined by (6.3) if we know solutions of (1.1). We obtain that properties of (6.1) can be determined using properties of (1.1). All equations (6.1) have partial integrability in the form of special solutions, but a number of these equations are integrable ones.

Substituting equations (k_s), (h_s) and (g_s) into (6.1) gives the new hierarchies of integrable equations

$$u_t+\frac{\partial}{\partial x}(ub^{n+1}(r))=0 \tag{k_n}$$

$$u_t+\frac{\partial}{\partial x}(uH_n(r))=0 \tag{h_n}$$

$$u_t+\frac{\partial}{\partial x}(uG_n(r))=0 \tag{g_n}$$

where

$$r=\frac{u_{xx}}{u}-\frac{u^2}{2}-\frac{3u_x^2}{2u^2}.$$

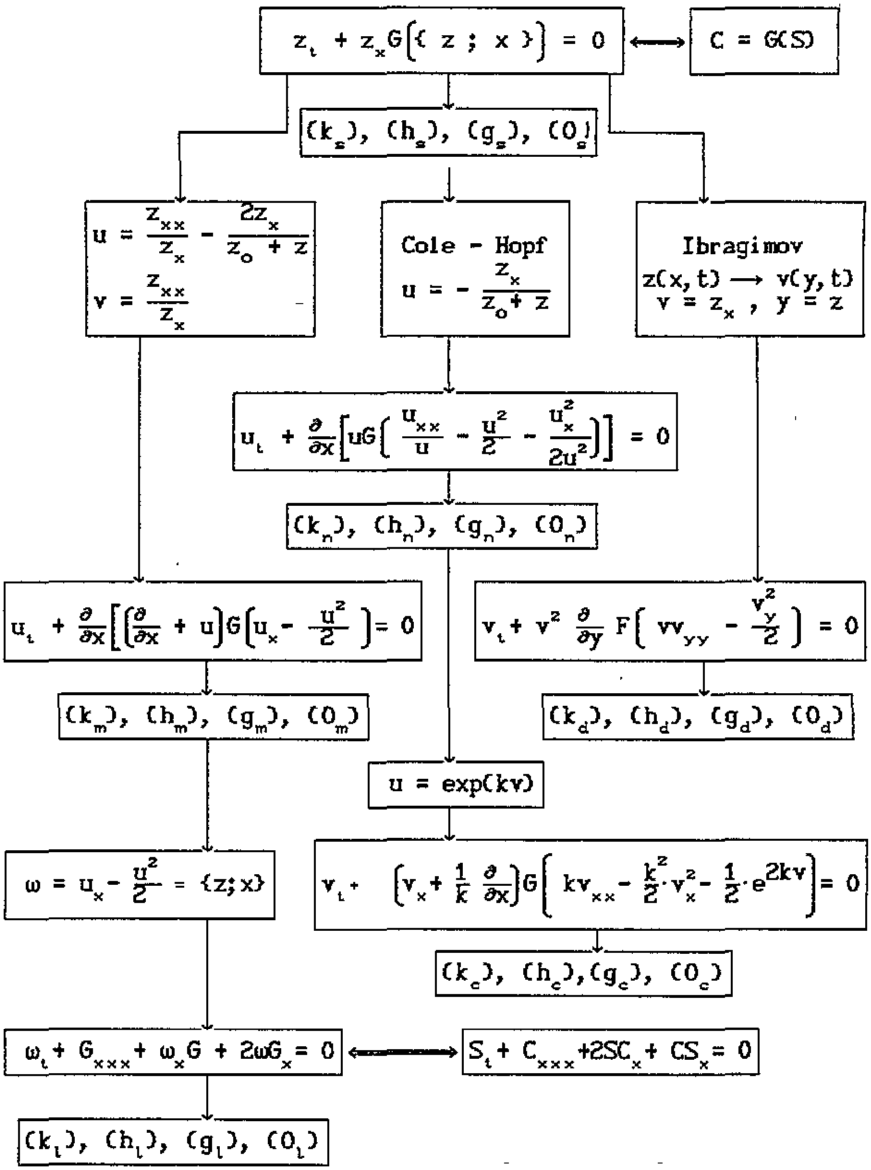

Figure 1. Families of evolution equations and hierarchies of integrable equations obtained from the simplest singular manifold equations.

These are given in figure 1.

We can also obtain the hierarchy of integrable equations

$$u_t + \frac{\partial}{\partial x}\left[u \sum_{n=0}^{N} \alpha_n b^{n+1}\left(\frac{u_{xx}}{u} - \frac{u^2}{2} - \frac{3u_x^2}{2u^2}\right)\right] = 0 \qquad (6.8)$$

from (6.1) taking into account the singular manifold equations (5.3).

It is clear that a lot of integrable partial differential equations can be obtained from (6.1) for such expressions of $G(\omega)$ which give integrable equations from (1.1).

In particular, if we take $G(\omega)=-\omega^{-2}$ in (2.4) we obtain from (6.1) the following integrable equation:

$$u_t=\frac{\partial}{\partial x}\left[u\left(\frac{u_{xx}}{u}-\frac{u^2}{2}-\frac{3u_x^2}{2u^2}\right)^{-2}\right]. \tag{6.9}$$

In the case $G(\omega)=-\omega^{-1/2}$ we have an integrable equation in the form

$$u_t=\frac{\partial}{\partial x}\left[u\left(\frac{u_{xx}}{2u}-\frac{u^2}{2}-\frac{3u_x^2}{2u^2}\right)^{-1/2}\right]. \tag{6.10}$$

Let us consider (7.8) at $N=0$ and $\alpha_0=1$, which takes the form

$$u_t+\frac{\partial}{\partial x}\left(u_{xx}-\frac{u^3}{2}-\frac{3u_x^2}{2u}\right)=0. \tag{6.11}$$

This equation is integrable because its solutions can be obtained by using solutions to the Krichever–Novikov equation (2.2). It has rational solutions which are found from (2.2) and (6.3).

Substituting

$$u=\mathrm{e}^{kv} \tag{6.12}$$

into (6.1) gives the family of equations

$$v_t+\left(v_x+\frac{1}{k}\frac{\partial}{\partial x}\right)G\left(kv_{xx}-\frac{k^2}{2}v_x^2-\frac{1}{2}\mathrm{e}^{2kv}\right)=0. \tag{6.13}$$

In the case

$$G=kv_{xx}-\frac{k^2}{2}v_x^2-\frac{1}{2}\mathrm{e}^{2kv}$$

we get from (6.13) the partial case of the Calogero–Degasperis–Focas equation [28, 29]

$$v_t-v_{xxx}-\frac{k^2}{2}v_x^3-\frac{3}{2}v_x\,\mathrm{e}^{2kv}=0. \tag{6.14}$$

Taking into account equations (k_s), (h_s) and (g_s) we obtain three hierarchies of equations from (6.13):

$$v_t+\left(v_x+\frac{1}{k}\frac{\partial}{\partial x}\right)b^{n+1}(p)=0 \tag{k_c}$$

$$v_t+\left(v_x+\frac{1}{k}\frac{\partial}{\partial x}\right)H_n(p)=0 \tag{h_c}$$

$$v_t+\left(v_x+\frac{1}{k}\frac{\partial}{\partial x}\right)G_n(p)=0 \tag{g_c}$$

where

$$p=kv_{xx}-\frac{k^2}{2}v_x^2-\frac{1}{2}\mathrm{e}^{2kv}.$$

These are given in figure 1.

One can also get the new hierarchy of equations if we use (6.13) and (5.3):

$$v_t+\left(v_x+\frac{1}{k}\frac{\partial}{\partial x}\right)\sum_{n=0}^{N}\alpha_n b^{n+1}\left(kv_{xx}-\frac{k^2}{2}v_x^2-\frac{1}{2}e^{2kv}\right)=0. \tag{6.15}$$

The solutions to the equations of families (6.13) and (6.15) can be obtained from the formula

$$v=\frac{1}{k}\ln\frac{\partial}{\partial x}\ln\frac{1}{z_0+z} \tag{6.16}$$

if we know the solutions of (1.1).

7. The family of Harry Dym equations

Consider the equations of the family having the form

$$v_t+v^2\frac{\partial}{\partial y}F\left(vv_{yy}-\frac{v_y^2}{2}\right)=0 \tag{7.1}$$

where $F(q)$ are smooth functions or operators of

$$q=vv_{yy}-\frac{v_y^2}{2}. \tag{7.2}$$

Assuming in (7.1) that

$$F=q=vv_{yy}-\frac{v_y^2}{2} \tag{7.3}$$

we obtain the Harry Dym equation [30]

$$v_t+v^3v_{yyy}=0. \tag{7.4}$$

The equations of family (7.1) are obtained from (1.1) using Ibragimov transformations [31]:

$$z(x,t)\to v(y,t) \qquad v=z_x \qquad y=z. \tag{7.5}$$

In fact, we have

$$z_{xx}=vv_y \tag{7.6}$$

$$z_{xxx}=v^2v_{yy}+vv_y^2 \tag{7.7}$$

$$z_{xt}=v_t+v_yz_t \tag{7.8}$$

taking into account transformation (7.5). Substituting (7.5)–(7.8) into (1.1) gives (7.1).

By assuming that $F=G(\omega)$ where $G(\omega)$ are determined by (2.3)–(2.5) we can obtain nonlinear integrable equations from (7.1).

In particular, we have the following integrable equations:

$$v_t+v^2\frac{\partial}{\partial y}\left(vv_{yy}-\frac{v_y^2}{2}\right)^{-2}=0 \tag{7.9}$$

$$v_t+2v^2\frac{\partial}{\partial y}\left(vv_{yy}-\frac{v_y^2}{2}\right)^{-1/2}=0. \tag{7.10}$$

We obtain three new hierarchies of equations from (7.1) if transformation (7.5) and the singular manifold equations (k_s), (h_s) and (g_s) are taken into account.

In particular, the hierarchy of (k_d) can be presented as

$$v_t+v^2\frac{\partial}{\partial y}\left[d^{n+1}\left(vv_{yy}-\frac{1}{2}v_y^2\right)\right]=0 \tag{7.11}$$

where the operator d^{n+1} has the form

$$\begin{aligned}&\frac{\partial}{\partial y}d^{n+1}=\frac{\partial}{\partial y}v\frac{\partial}{\partial y}v\frac{\partial}{\partial y}d^n+2q\frac{\partial}{\partial y}d^n+2q_yd^n\\ &d^0=1 \qquad d^1=q=vv_{yy}-\tfrac{1}{2}v_y^2.\end{aligned} \tag{7.12}$$

The equations of family (7.1) have special solutions because of the exact solutions of (1.1).

8. Conclusion

We have presented families of evolution equations and hierarchies of integrable equations which were obtained from the simplest singular manifold equations (1.1). These families and hierarchies are given in figure 1. We have denoted hierarchies of singular manifold equations having the Painleve property as (k_s), (h_s) and (g_s). Other integrable singular manifold equations are grouped with (O_S). Taking into account transformations (3.2), (3.3), (3.8), (6.3), (6.12) and (7.5), we have found 15 hierarches of integrable equations (k_m), (k_l), (k_n), (k_c), (k_d), (h_m), (h_l), (h_n), (h_c), (h_d), (g_m), (g_l), (g_n), (g_c) and (g_d). All these equations are integrable because their solutions can be obtained from equations (k_s), (h_c) and (g_l), which have the Painleve property. We have also obtained a number of other integrable equations, (O_m), (O_l), (O_n), (O_c) and (O_d), using (2.3), (2.4) and (2.5).

Note that the number of hierarchies of integrable equations in figure 1 is well known. In particular, Weiss has considered in detail the hierarchies (k_l), (h_l), (g_l) and (k_m) [12].

We can se from figure 1 that the equations (k_m), (h_m), (g_m), (k_l), (h_l), (g_l), (k_n), (h_n) and (g_n) have the Painleve property for partial differential equations, but (k_c), (h_c), (g_c), (k_d), (h_d), (g_d) and other equations do not have this property.

Acknowledgments

I am grateful to my pupils G Homishin, V Nikitin and Jane Zargaryan for their help.

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