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Truncated expansions and nonlinear integrable partial differential equations

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The families of nonlinear equations having Lax pairs have been found starting with the truncated expansions. The properties of these equations are studied. The link is shown between these equations and the Lax equations.

1. Introduction

The Painlevé test is one of the impressive methods for the investigation of differential equations [1]. In recent years it has been successfully applied both to integrable equations [1–5] and nonintegrable ones [6–10]. The definition of the Painlevé property of partial differential equations and the discussion of many importment peculiarities relating to the Painlevé property are given in ref. [11]. When an equation has the Painlevé property one can construct auto-Bäcklund transformations by truncating an expansion of the solution about the movable singularity manifold at the constant level term. It is easy to verify that the so-called truncated expansions (auto-Bäcklund transformations) play a special role in the investigation of the equations.

Recently [12] we used the truncated expansions to construct the family of partial differential equations with solutions having movable first-order singularities. The family of integrable equations that was obtained in ref. [12] has the form

$$u_t = \frac{\partial}{\partial x}\left(\sum_{n=0}^{N} \alpha_n (\partial/\partial x + u)^n u\right),$$

$$\alpha_n = \alpha_n(x, t) . \tag{1.1}$$

Solutions of eqs. (1.1) can be obtained by means of the formula

$$u = z_x W(z) , \quad W(z) = \frac{1}{z_0 + z} \tag{1.2}$$

(where z_0 is constant) across solutions of the following linear equations,

$$z_t = \sum_{n=0}^{N} \alpha_n z_{n+1,x} , \quad z_{k,x} = \frac{\partial^k z}{\partial x^k} . \tag{1.3}$$

To demonstrate this explicitly we can write

$$u_t - \frac{\partial}{\partial x} \sum_{n=0}^{N} \alpha_n (\partial/\partial x + u)^n u$$

$$= \frac{\partial}{\partial x}\left[W\left(z_t - \sum_{n=0}^{N} \alpha_n z_{n+1,x}\right)\right] \tag{1.4}$$

taking into account the expression

$$W z_{k,x} = (\partial/\partial x + u)^{k-1} u . \tag{1.5}$$

Apparently the family of equations (1.1) does not contain the modified Korteweg–de Vries (MKdV) equation. However, it is well known that the MKdV equation has solutions with movable first-order singularities [1].

Using the truncated expansions we want to obtain the family of integrable equations more general than (1.1) in this Letter.

The outline of this Letter is as follows: in section 2 we describe the families of equations studied and give examples of physical applications, we prove also two lemmas, which give the basic idea of our approach; in section 3 we discuss the properties of one of the families; in section 4 the Bäcklund transformations of eqs. (1.1) are considered and in section 5 we deal with spectral and auxiliary problems for our equations; in section 6 we find the link between our equations and the Lax equations.

2. The family of equations studied

In this Letter we are going to study the following family of equations,

$$u_t+\frac{\partial}{\partial x}(Q_x+2uQ)=0\,, \tag{2.1}$$

where a is constant, $Q(u,\, u_x,\, ...,\, x,\, t)$ is a smooth function of u, u_x, ..., x and t.

Among the basic equations of family (2.1) we have the Burgers equation for $Q=u$ [13–15], the modified KdV equation for $J=u_x-u^2$ [16], the equation of filtration of a liquid through a porous medium for $Q=u^m$, $m>0$ [17–19] and the nonlinear equation of diffusion for $Q=u^{-1}$ [20,21]. Later we shall show that the equations of family (2.1) have inverse problems.

However, at the beginning we consider the family of equations

$$u_t+\frac{\partial}{\partial x}[(\partial/\partial x+2u)G(u_x-u^2)]=0\,, \tag{2.2}$$

where $G(u_x-u^2)$ is an operator or a smooth function of u_x-u^2. Obviously, the family of equations (2.2) contains the modified KdV equation for $G=u_x-u^2$.

For example, we can take the following expressions for G in (2.2),

$$G_1=\sum_{m=0}^{M}\gamma_m\frac{\partial^m}{\partial x^m}(u_x-u^2)\,, \tag{2.3}$$

$$G_2=\sum_{k=1}^{K}\beta_k(u_x-u^2)^k\,. \tag{2.4}$$

Let us seek for the solutions of eqs. (2.2) by using the truncated expansion in the form

$$u=-z_xW(z)+v\,,\quad W(z)=\frac{1}{z_0+z}\,. \tag{2.5}$$

The following lemmas give the basic idea of our approach.

Lemma 2.1. Let (2.5) be the transformation with

$$v=z_{xx}/2z_x\,, \tag{2.6}$$

then

$$I=u_x-u^2=v_x-v^2 \tag{2.7}$$

is the invariant under transformation (2.5).

Proof. We have

$$u_x-u^2=-(z_{xx}-2vz_x)w+v_x-v^2 \tag{2.8}$$

after differentiation of (2.5) with respect to x and adding u^2 to both sides of the equality. After that we obtain (2.7) taking into account (2.6).

Remark 2.1. Substituting

$$u=-\frac{z_x}{z_0+z}+\frac{z_{xx}}{2z_x}\,,\quad v=\frac{z_{xx}}{2z_x} \tag{2.9}$$

into (2.7) gives

$$I_1=-\frac{3z_{xx}^2}{4z_x^2}+\frac{z_{xxx}}{2z_x}=\tfrac{1}{2}\{z;x\}\,, \tag{2.10}$$

where $\{z;\,x\}$ is the well-known Schwarzian derivative [2].

Lemma 2.2. Let (2.5) be the transformation with $v=z_{xx}/2z_x$, then the equations of family (2.2) are presented in the form

$$\begin{aligned}&u_t+\frac{\partial}{\partial x}[(\partial/\partial x+2u)G(u_x-u^2)]\\&\quad=v_t+\frac{\partial}{\partial x}[(\partial/\partial x+2v)G(v_x-v^2)]\\&\quad+\frac{\partial}{\partial x}[W(z_t+2z_xG(\tfrac{1}{2}\{z;x\}))]\,.\end{aligned} \tag{2.11}$$

Proof. We get (2.11) substituting (2.5) into eqs. (2.2) and taking into account invariant (2.7).

We will use expression (2.11) for investigating eqs. (2.2).

Remark 2.2. We can use an invariant more general than (2.7), namely

$$u_x-u^2+f(x,t)=v_x-v^2+f(x,t)\,,$$

where $f(x,t)$ is some function. In this case the family of integrable equations has coefficients depending on x and t.

3. The properties of eqs. (2.2)

Let us show that equations of family (2.2) have Bäcklund transformations analogous to the modified KdV equation.

It is convenient for us to do this using a few theorems.

Theorem 3.1. Let v be a solution of the set of equations

$$z_{xx}=2vz_x\,, \tag{3.1}$$

$$z_t+2z_xG(v_x-v^2)+q(t)(z+z_0)=0\,, \tag{3.2}$$

then eqs. (2.2) are the compatibility condition of eqs. (3.1), (3.2). Here $q(t)$ is an arbitrary function of t.

Proof. We get (2.2) taking into account

$$(z_{xx})_t=(z_t)_{xx}\,. \tag{3.3}$$

Theorem 3.2. Let $z(x,t)$ be a solution of the equation

$$z_t=2z_xG(\tfrac{1}{2}\{z;x\})\,, \tag{3.4}$$

then the expressions

$$u=-\frac{z_x}{z_0+z}+\frac{z_{xx}}{2z_x}\,,\quad v=\frac{z_{xx}}{2z_x} \tag{3.5}$$

allow one to find solution (2.2).

Proof. This follows from (2.11) if we show that

$$v_t+\frac{\partial}{\partial x}\left[G_x(v_x-v^2)+2vG(v_x-v^2)\right]=0\,, \tag{3.6}$$

with $z(x,t)$ satisfying eqs. (3.4).

We get

$$\frac{\partial}{\partial x}\left(\frac{1}{2z_x}\frac{\partial}{\partial x}\left[z_t+2z_xJ(\tfrac{1}{2}\{z;x\})\right]\right)=0 \tag{3.7}$$

substituting $v=z_{xx}/2z_x$ into (3.6).

After this we have from (2.11) the following expression,

$$u_t+\frac{\partial}{\partial x}\left[(\partial/\partial x+2u)G(u_x-u^2)\right]$$
$$=-\frac{\partial}{\partial x}\left(W-\frac{1}{2z_x}\frac{\partial}{\partial x}\right)\left[z_t+2z_xG(\tfrac{1}{2}\{z;x\})\right]\,, \tag{3.8}$$

which proves theorem 3.2.

Remark 3.1. Let us denote the function of the velocity [6]

$$C=-z_t/z_x \tag{3.9}$$

and the Schwarzian derivative [2]

$$S=\frac{z_{xxx}}{z_x}-\frac{3}{2}\frac{z_{xx}^2}{z_x^2}\,, \tag{3.10}$$

then eqs. (3.4) can be written as

$$C=2G(\tfrac{1}{2}S)\,. \tag{3.11}$$

These functions C and S are invariant under the Möbius group [3].

At first glance it would seem that all equations of family (2.2) have the Painlevé property. In fact, we will have three arbitrary functions in the expansion of the solution if we take $G=u_x-u^2$, as follows from (3.8). Other equations of family (2.2) can have a derivative of u_x-u^2 with respect to x or t and we obtain the full complement of arbitrary functions substituting (2.5) into (2.2) because of eqs. (2.7) and (2.8). But the Painlevé property requires that all movable singularities be single-valued [4,11]. This is not always the case for equations of family (2.2). For example if we take

$$G=\frac{\partial}{\partial x}(u_x-u^2)\,, \tag{3.12}$$

we will get an equation from (2.2) that does not have the Painlevé property [4,11].

Among the equations of family (2.2) there are equations both having and not having the Painlevé property. It depends on the form of eqs. (3.4).

4. The Bäcklund transformations for equations of family (1.1)

Now we consider additional properties of eqs. (1.1) in comparison with the ones studied in ref. [12].

We will seek for solutions of eqs. (1.1) in the form

$$u=z_x W+v\,, \quad W=\frac{1}{z_0+z}\,. \tag{4.1}$$

Let the operator L^k be defined by the following expression,

$$L^k u=(\partial/\partial x+u)^k u\,. \tag{4.2}$$

We need the following statements now.

Lemma 4.1. Let (4.1) be the transformation, then

$$L^k u=L^k v+P_k W \quad (k=0, ..., N)\,, \tag{4.3}$$

$$P_k=z_x L^{k-1}v+(\partial/\partial x+v)P_{k-1}\,, \quad (k=1, ..., N)\,, \tag{4.4}$$

$$P_0=z_x\,.$$

Proof. Let us apply the method of mathematical induction. We have

$$L^0 u=u\,, \quad L^0 v=v\,, \quad P_0=z_x$$

by definition and we get (4.3) for $k=0$.

Let us suggest that $m=k$ in eq. (4.3), then

$$L^{m+1}u=\frac{\partial}{\partial x}L^m u+uL^m u$$

is transformed to eq. (4.3) for $k=m+1$.

Lemma 4.2. Let (4.1) be the transformation, then eqs. (1.1) are presented in the form

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\sum_{n=0}^{N}\alpha_n L^n u=\frac{\partial v}{\partial t}-\frac{\partial}{\partial x}\sum_{n=0}^{N}\alpha_n L^n v$$
$$+\frac{\partial}{\partial x}\left[W\left(z_t-\sum_{n=0}^{N}\alpha_n P_n\right)\right]. \tag{4.5}$$

Proof. We obtain (4.5) taking into account eqs. (4.3) and (4.4).

Consequence 4.1. Expression (4.4) is a linear function of the derivative $z_{l,x}$ ($l=1, ..., k+1$) for P_k.

Proof. We have $P_0=z_x$, $P_1=2z_x v+z_{xx}$ and get consequence 4.1 applying the method of mathematical induction.

We will study the family of equations (1.1) using (4.5).

Theorem 4.1. Let v be a solution of (1.1), then u gives a solution of (1.1) from (4.1) again, if $z(x, t)$ is a solution of the linear equation

$$z_t=\sum_{n=0}^{N}\alpha_n P_n\,. \tag{4.6}$$

Proof. This follows from expression (4.5) directly.

Theorem 4.2. Let $z(x, t)$ be a solution of (1.1), then

$$u=z+z_x W(z) \tag{4.7}$$

will be a solution of eq. (1.1) too.

Proof. If we assume that $v=z$ and $P_0=z_x$, then we will get

$$P_1=\frac{\partial}{\partial x}(z_x+z^2)=\frac{\partial}{\partial x}L^1 z \tag{4.8}$$

from (4.4) for α_n constants. We have

$$P_n=\frac{\partial}{\partial x}L^n z \quad (n=1, ..., N) \tag{4.9}$$

using the method of mathematical induction.

We find from (4.5)

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\sum_{n=0}^{N}\alpha_n L^n u$$
$$=\left(1+\frac{\partial}{\partial x}W\right)\left(z_t-\frac{\partial}{\partial x}\sum_{n=0}^{N}\alpha_n L^n z\right) \tag{4.10}$$

taking into account (4.9).

This proves theorem 4.2.

Theorem 4.3. Equations (1.1) have the Painlevé property.

Proof. This follows from (4.5).

5. Isospectral problems for the equations of family (2.1)

Above we obtained that some equations of families (1.1), (2.2) have the Painlevé property. We have to remark that all these equations are contained in the family of equations (2.1). It is known, that if an equation possesses the Painlevé property it is indeed integrable, i.e. the Painlevé test is a sufficient condition for integrability [7]. Below we attempt to find isospectral problems for our equations.

Consider the spectral and auxiliary problems [16]

$$\psi_x = U\psi , \tag{5.1}$$

$$\psi_t = V\psi , \tag{5.2}$$

where

$$\psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \tag{5.3}$$

and the matrices U and V are presented in the form

$$U=\begin{pmatrix}-\lambda & u-\lambda\\ u+\lambda & \lambda\end{pmatrix}, \quad V=\begin{pmatrix}A & B+A\\ B-A & -A\end{pmatrix}. \tag{5.4}$$

Then (5.1), (5.2) may be presented in the form of the zero-curvature equation

$$U_t - V_x + UV - VU = 0 . \tag{5.5}$$

We find the conditions from (5.5) taking into account (5.4),

$$\lambda_t = 0 , \tag{5.6}$$

$$A_x + 2uA = -2\lambda B , \tag{5.7}$$

$$u_t - B_x = 0 . \tag{5.8}$$

We obtain that (5.1), (5.2) are the spectral problems for the equations of family (2.1), if the matrices U and V have the form

$$U=\begin{pmatrix}-\lambda & u-\lambda\\ u+\lambda & \lambda\end{pmatrix}, \tag{5.9}$$

$$V=\begin{pmatrix}2\lambda Q & 2\lambda Q-(Q_x+2uQ)\\ -2\lambda Q-(Q_x+2uQ) & -2\lambda Q\end{pmatrix}, \tag{5.10}$$

assuming

$$B = -(Q_x + 2uQ) , \tag{5.11}$$

$$A = 2\lambda Q , \tag{5.12}$$

in (5.7), (5.8).

This means that eqs. (1.1), (2.2) have isospectral problems (5.1), (5.2) and (5.9), (5.10) too.

6. Link between the equations of family (2.2) and the Lax equations

It is well known that the Miura transformation [22]

$$\omega = u_x - u^2 \tag{6.1}$$

allows one to find solutions of the Korteweg–de Vries equation using solutions of the modified KdV equation, which follows from the equality

$$\omega_t - 6\omega\omega_x + \omega_{xxx} = -(\partial/\partial x - 2u)(u_t - 6u^2 u_x + u_{xxx}) . \tag{6.2}$$

Let us show that the Miura transformation (6.1) is the link between solutions of eqs. (2.2) and the solutions of the Lax equations

$$\omega_t + G_{xxx} + 4\omega G_x + 2\omega_x G = 0 , \quad G = G(\omega, \omega_x, ...) . \tag{6.3}$$

Theorem 6.1. Let u be a solution of eq. (2.2), then (6.1) gives the solution ω of eq. (6.3).

Proof. The statement follows from the expression

$$\omega_t + G_{xxx} + 4\omega G_x + 2\omega_x G = (-2u + \partial/\partial x) \times\left(u_t + \frac{\partial}{\partial x}\,[G_x + 2uG(u_x - u^2)]\right). \tag{6.4}$$

We have

$$\omega = \tfrac{1}{2}\{z; x\} \tag{6.5}$$

taking into account invariant (2.7).

Theorem 6.2. Let $z(x, t)$ be a solution of eq. (3.4), then (6.5) gives the solution of eqs. (6.3).

Proof. The statement follows from the expression

$$\omega_t + G_{xxx} + 4\omega G_x + 2\omega_x G$$

$$= \left(\frac{\partial}{\partial x} + 2z_x W - \frac{z_{xx}}{z_x}\right)\frac{\partial}{\partial x}\left(W - \frac{1}{2z_x}\frac{\partial}{\partial x}\right)$$

$$\times [z_t + 2z_x G(\tfrac{1}{2}\{z; x\})] \tag{6.6}$$

Among the equations of family (6.3) there are also equations both possessing and not possessing the Painlevé property. This also depends on the form of eqs. (3.4).

Substituting $\omega + \lambda$ for ω into eq. (6.3) we obtain the Lax equations

$$\omega_t + G_{xxx} + 4\omega G_x + 2\omega_x G + 4\lambda G_x = 0 . \tag{6.7}$$

Theorem 6.3. Let ω be a solution of the linear set of equations

$$\varphi_{xx} + (\omega + \lambda)\varphi = 0 , \tag{6.8}$$

$$\varphi_t + 2\varphi_x G(\omega) - \varphi G_x(\omega) = 0 , \tag{6.9}$$

then eqs. (6.7) are the compatibility condition of the set (6.8), (6.9).

Proof. We get the set of equations (6.8), (6.9) substituting $u = -\varphi_x/\varphi$ into (6.1) and (2.2).

We obtain the proof of this theorem from the compatibility condition

$$(\varphi_{xx})_t = (\varphi_t)_{xx} . \tag{6.10}$$

From eq. (6.7) we can get the hierarchy of the KdV equation using the expansion [16]

$$G = G_0\lambda^n + ... + G_n . \tag{6.11}$$

Assuming $\omega = \frac{1}{2}S$ we have from (3.4)

$$G(\tfrac{1}{2}S) = \tfrac{1}{2}C . \tag{6.12}$$

Substitution of eq. (6.12) and $\omega = \frac{1}{2}S$ into eq. (6.3) gives the equation

$$S_t + C_{xxx} + 2SC_x + CS_x = 0 . \tag{6.13}$$

This equation is the compatibility condition of the functions C and S [6].

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