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Singular manifold equations and exact solutions for some nonlinear partial differential equations

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The singular manifold equations are used to seek the solutions of nonlinear partial differential equations. Special solutions of these equations are obtained. Integrable equations having the Painlevé property are presented.

1. Introduction

The application of the Painlevé test to investigate partial differential equations is well-known (see for example refs. [1-6]). The definition of the Painlevé property of partial differential equations and a discussion of important peculiarities of it are given in ref. [2].

Recently we made an attempt to construct the partial differential equations having the Painlevé property [7]. As a starting point for our purpose we have used the truncated expansion of the solutions. In this approach we have supposed that some partial differential equations of evolution type

$$u_t=E(u, u_x, u_{xx}, ...) \tag{1.1}$$

have solutions in the form of the truncated expansions (auto-Bäcklund transformations)

$$u=-Z_x W+v\,,$$

$$W=\frac{1}{Z_0+Z},\quad Z=Z(x,t)\,, \tag{1.2}$$

where $u(x, t)$ and $v(x, t)$ are solutions of eq. (1.1), $Z(x, t)$ is a new function and Z_0 is an arbitrary constant.

Assuming $v=0$ in eq. (1.2) we have identified the following family of integrable partial differential equations,

$$u=\frac{\partial}{\partial x}\left(\sum_{n=1}^{N}\alpha_n(\partial/\partial x-u)^n u\right), \tag{1.3}$$

where $\alpha_n(x, t)$ are coefficients.

Solutions of eqs. (1.3) can be obtained by the formula

$$u=-\frac{Z_x}{Z_0+Z}, \tag{1.4}$$

if $Z(x, t)$ satisfies the linear equation

$$Z_t=\sum_{n=1}^{N}\alpha_n Z_{n+1,x},\quad Z_{n,x}=\frac{\partial^n Z}{\partial x^n}. \tag{1.5}$$

This follows from the equality

$$u_t-\frac{\partial}{\partial x}\left(\sum_{n=1}^{N}\alpha_n(\partial/\partial x-u)^n u\right)$$

$$=-\frac{\partial}{\partial x}\left[W\left(Z_t-\sum_{n=1}^{N}\alpha_n Z_{n+1,x}\right)\right]. \tag{1.6}$$

Elsewhere [8] we have considered the following family of partial differential equations,

$$u_t+\frac{\partial}{\partial x}\left[(\partial/\partial x+2u)G(u_x-u^2)\right]=0\,, \tag{1.7}$$

where $G(u_x-u^2)$ is a smooth function or operator of u_x-u^2.

Among the basic equations of family (1.7) we have the modified Korteweg–de Vries equation for $G=u_x-u^2$.

Assuming in eq. (1.2)

$$v=\frac{Z_{xx}}{2Z_x}, \qquad (1.8)$$

we get the following equality,

$$\omega=u_x-u^2=v_x-v^2=\tfrac{1}{2}\{Z;x\}, \qquad (1.9)$$

where $\{Z;x\}$ is the Schwarzian derivative [9]

$$\{Z;x\}=\frac{Z_{xxx}}{Z_x}-\frac{3}{2}\frac{Z_{xx}^2}{Z_x^2}. \qquad (1.10)$$

The solution of eqs. (1.7) can be obtained by using the formulas

$$u=-\frac{Z_x}{Z_0+Z}+\frac{Z_{xx}}{2Z_x},\quad v=\frac{Z_{xx}}{2Z_x}, \qquad (1.11)$$

if $Z(x,t)$ is the solution of the following equations,

$$Z_t+2Z_xG(\tfrac{1}{2}\{Z;x\})=0. \qquad (1.12)$$

This follows from the equalities

$$u_t+\frac{\partial}{\partial x}[(\partial/\partial x+2u)G(u_x-u^2)]$$
$$=-\frac{\partial}{\partial x}\left(W-\frac{1}{2Z_x}\frac{\partial}{\partial x}\right)[Z_t+2Z_xG(\tfrac{1}{2}\{Z;x\})], \qquad (1.13)$$

$$v_t+\frac{\partial}{\partial x}[(\partial/\partial x+2v)G(v_x-v^2)]$$
$$=\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{Z_x}\frac{\partial}{\partial x}[Z_t+2Z_xG(\tfrac{1}{2}\{Z;x\})]\right). \qquad (1.14)$$

Let us take the well-known family of Lax equations [10],

$$\omega_t+G_{xxx}+2\omega_xG+4\omega G_x=0,$$
$$G=G(\omega,\omega_x,...), \qquad (1.15)$$

and the Miura transformation [11],

$$\omega=u_x-u^2, \qquad (1.16)$$

then substitution of (1.16) into (1.15) shows that

$$\omega_t+G_{xxx}+2\omega_xG+4\omega G_x=(\partial/\partial x-2u)$$
$$\times\left(u_t+\frac{\partial}{\partial x}(\partial/\partial x+2u)G(u_x-u^2)\right). \qquad (1.17)$$

Clearly if u is a solution of (1.17), then ω, defined by eq. (1.16), is a solution of eq. (1.15).

Taking into account (1.11), (1.13) and (1.17) we get

$$\omega_t+G_{xxx}+2\omega_xG+4\omega G_x$$
$$=\left(2Z_xW-\frac{Z_{xx}}{Z_x}-\frac{\partial}{\partial x}\right)\frac{\partial}{\partial x}\left(W-\frac{1}{2Z_x}\frac{\partial}{\partial x}\right)$$
$$\times[Z_t+2Z_xG(\tfrac{1}{2}\{Z;x\})]. \qquad (1.18)$$

Note that eq. (1.12) can be presented in the form

$$C=2G(\tfrac{1}{2}S), \qquad (1.19)$$

if we denote [4,5]

$$C=-\frac{Z_t}{Z_x},\quad \omega=\tfrac{1}{2}S,\quad S=\{Z;x\}. \qquad (1.20)$$

Substitution of (1.19) and (1.20) into (1.18) gives

$$S_t+2G_{xxx}+2GS_x+4SG_x=2\left(\frac{\partial}{\partial x}+\frac{Z_{xx}}{Z_x}-2Z_xW\right)$$
$$\times\frac{\partial}{\partial x}\left(W-\frac{1}{2Z_x}\frac{\partial}{\partial x}\right)Z_x[C-2G(\tfrac{1}{2}S)]. \qquad (1.21)$$

The functions C and S are known to be invariant under the Möbius group [9].

Let eq. (1.19) hold, then we obtain from (1.21)

$$S_t+C_{xxx}+CS_x+2SC_x=0. \qquad (1.22)$$

Equality (1.21) also explains why the Painlevé–Bäcklund equations that arise on application of the singular manifold method are invariant under the Möbius group [4,9].

The outline of this Letter is as follows: in section 2 we discuss integrable equations of family (1.7); in section 3 we seek special solutions of eqs. (1.7) and some equations close to them; in sections 4 and 5 our approach is generalized to some equations in 2+1 dimension.

2. Integrable equations of family (1.7)

The equations of family (1.7) seem to have the Painlevé property. In fact, we get three arbitrary functions in the expansion of the solution of eqs. (1.7) at $G=u_x-u^2$. Other equations of family (1.7) can have derivatives of u_x-u^2 and as a consequence additional arbitrary functions because of the invariant (1.9). But the Painlevé property requires that all movable singularities of the solution be single-valued [2]. This breaks down for a number of equations of family (1.7).

For example if we take in (1.7)

$$G=\frac{\partial}{\partial x}(u_x-u^2)\,, \tag{2.1}$$

we will get an equation not having the Painlevé property [12]. But we will show later that this equation has special solutions.

To investigate the Painlevé property for eqs. (1.7) we can use eqs. (1.12) for the function $Z(x, t)$.

For G a function of ω these equations were considered in ref. [13] using the properties of symmetries. It turned out that eqs. (1.12) are integrable, when G takes the form

$$G(\omega)=C_1+C_2\omega\,, \tag{2.2}$$

$$G(\omega)=C_1-\tfrac{1}{2}(C_2\omega+C_3)^{-2}\,, \tag{2.3}$$

$$G(\omega)=C_1-\frac{C_2+2C_3\omega}{2\sqrt{C_4+C_2\omega+C_3\omega^2}}\,, \tag{2.4}$$

where C_1, C_2, C_3 and C_4 are arbitrary constants.

Substitution of (2.2) into (1.7) gives the modified Korteweg–de Vries equation.

Assuming $C_1=0$, $C_3=0$, $C_2=\frac{1}{2}$ in (2.3) we get the following integrable equations from (1.7) and (1.15),

$$u_t=\frac{\partial}{\partial x}\left[(\partial/\partial x+2u)(u_x-u^2)^{-2}\right]\,, \tag{2.5}$$

$$\omega_t+(\omega^{-2})_{xxx}+2\omega_x\omega^{-2}+4\omega(\omega^{-2})_x=0\,. \tag{2.6}$$

Taking into account expressions (2.3), (2.4) we can find also other integrable equations from (1.7) and (1.15). To obtain solutions of these equations it is necessary to use eqs. (1.9), (1.11) and solutions of eqs. (1.12).

Weiss [12] has studied eqs. (1.12) on the basis that these equations will identically possess the Painlevé property when there exists a transformation

$$Z_x=\varphi_x^m\,, \tag{2.7}$$

where m is rational and negative and φ also satisfies an equation (1.12). He found some forms $G(\omega)$ that give integrable partial differential equations of families (1.7) and (1.15). In particular eqs. (1.7) and (1.15) have the Painlevé property when the $G(\omega)$ are of the form

$$G(\omega)=\tfrac{1}{2}\omega_{xx}+\tfrac{3}{4}\omega^2\,, \tag{2.8}$$

$$G(\omega)=\tfrac{1}{2}\omega_{xx}+\tfrac{1}{8}\omega^2\,, \tag{2.9}$$

$$G(\omega)=\tfrac{1}{2}\omega_{xx}+2\omega^2\,. \tag{2.10}$$

In case (2.8), eq. (1.15) is the first higher-order KdV equation. Expressions (2.9) and (2.10) give the Kupershmidt and Caudrey–Dodd–Gibbon equations.

The dual invariance of the integrable equations (1.12) allows one to find rational solutions of eqs. (1.7) and (1.15) using the discrete symmetries as described in ref. [12].

Weiss has obtained the following rational solutions,

$$Z_1=x\,, \tag{2.11}$$

$$Z_2=x^3+12t\,, \tag{2.12}$$

$$Z_3=(x^6+60x^3t+C_1x-720\,t^2)/x \tag{2.13}$$

of the equation

$$Z_t+Z_{xxx}-\frac{3Z_{xx}^2}{2Z_x}=0\,, \tag{2.14}$$

using the iterative formula [12]

$$\frac{\partial Z_{n+1}}{\partial x}=Z_n^2\left(\frac{\partial Z_n}{\partial x}\right)^{-1}\quad(n=1,2,\ldots)\,. \tag{2.15}$$

Substituting (2.11)–(2.13) into eqs. (1.11) we find rational solutions of the modified Korteweg–de Vries equation.

Using this approach one can seek rational solutions of the other modified equations as well.

3. Special solutions of eqs. (1.7)

It is convenient to use the singular manifold equations (1.12) for obtaining special solutions of eqs. (1.7).

Let us consider the solitary wave solution of these equations,

$$Z(x,t)=Z(\xi),\quad \xi=x-C_0 t\,. \tag{3.1}$$

We then get

$$G(\tfrac{1}{2}\{Z;\xi\})=\tfrac{1}{2}C_0 \tag{3.2}$$

from (1.12).

Assuming that

$$\{Z;\xi\}=-2k^2\,, \tag{3.3}$$

where k is an arbitrary constant, we have

$$C_0=2G(-k^2)\,. \tag{3.4}$$

Solving eq. (3.3) we get

$$v=\frac{Z_{\xi\xi}}{2Z_\xi}=-k\tanh(k\xi+\varphi_0)\,, \tag{3.5}$$

$$Z(\xi)=b-a\tanh(k\xi+\varphi_0)\,, \tag{3.6}$$

where φ_0, a and b are arbitrary constants.

Substitution of (3.5) and (3.6) into (1.11) gives solutions of eqs. (1.7) in the form

$$u=\frac{ak\tanh(k\xi+\varphi_0)}{a\tanh(k\xi+\varphi_0)-b}-k\tanh(k\xi+\varphi_0)\,. \tag{3.7}$$

In the case of the concrete function $G(\omega)$ one can find other solutions of eqs. (1.7). For example if we take for $G(\omega)$ eq. (2.1) then we will get the solution of eq. (1.7) in terms of the standard Airy function.

Note that using this approach one can obtain a solution of equations more general than eqs. (1.7).

Let us consider the following equations,

$$u_t+\frac{\partial}{\partial x}[(\partial/\partial x+2u)G(u_x-u^2)]$$
$$+Q(u_x-u^2)=0\,, \tag{3.8}$$

where Q is a smooth function or operator of u_x-u^2.

One can seek solutions of these equations as solutions of the overdetermined set of equations

$$Z_t+2Z_xG(\tfrac{1}{2}\{Z;x\})=0\,, \tag{3.9}$$

$$Q(\tfrac{1}{2}\{Z;x\})=0\,. \tag{3.10}$$

For example let us find a solution of the generalized Burgers–Korteweg–de Vries equation,

$$u_t+2\alpha uu_x-6u^2u_x+u_{xxx}=\alpha u_{xx}\,, \tag{3.11}$$

where α is an arbitrary constant.

This equation is obtained from (3.8) for $G=u_x-u^2$ and $Q=-\alpha(\partial/\partial x)(u_x-u^2)$. The overdetermined set of equations (3.9), (3.10) in this case has the form

$$Z_t+Z_{xxx}-\frac{3Z_{xx}^2}{2Z_x}=0\,, \tag{3.12}$$

$$\frac{\partial}{\partial x}\{Z;x\}=0\,. \tag{3.13}$$

The solution of this set of equations is expressed in terms of function (3.7).

For a second example let us take the equation

$$u_t+\frac{\partial}{\partial x}[(u^2)_x-2u^3]=0\,. \tag{3.14}$$

This equation is frequently referred to as the porous media equation [14–16]. It coincides with eq. (3.8) if we suppose $G=u_x-u^2$ and $Q=-(\partial^2/\partial x^2)(u_x-u^2)$.

The overdetermined set of equations (3.9), (3.10) in this case has the form

$$Z_t+Z_{xxx}-\frac{3Z_{xx}^2}{2Z_x}=0\,, \tag{3.15}$$

$$\frac{\partial^2}{\partial x^2}\{Z;x\}=0\,. \tag{3.16}$$

Let us look for solutions of the set (3.15), (3.16) as the self-similar solutions

$$Z(x,t)=Z(\vartheta),\quad \vartheta=x/t^{1/3}\,. \tag{3.17}$$

We then get from (3.15) and (3.16)

$$\{Z;\vartheta\}=\tfrac{1}{3}\vartheta\,. \tag{3.18}$$

The solution of this equation takes the form

$$Z(\vartheta)=C_1\int_0^\vartheta \Psi(\xi)^{-2}\,\mathrm{d}\xi\,, \tag{3.19}$$

where

$$\Psi(\vartheta)=C_2\mathrm{Ai}(\tfrac{1}{3}\vartheta)+C_3\mathrm{Bi}(\tfrac{1}{2}\vartheta)\,. \tag{3.20}$$

C_1, C_2, C_3 are arbitrary constants, $\mathrm{Ai}(y)$ and $\mathrm{Bi}(y)$ are the Airy functions.

The solution of eq. (3.14) can be presented in the form

$$u=\frac{C_1}{t^{1/3}}\left[\Psi^2\left(C_4-\int_0^{\vartheta}\Psi(\xi)^{-2}\,d\xi\right)\right]^{-1}$$
$$-\frac{1}{t^{1/3}}\frac{d}{d\vartheta}\ln\Psi, \tag{3.21}$$

where C_1 and C_4 are arbitrary constants.

4. Integrable equations in 2+1 dimension

It is easy to see that the family of equations (1.7) can be generalized for equations in 2+1 dimension. This goal can be achieved if we assume in (1.2) that $Z=Z(x, y, t)$, then we will have $v=v(x, y, t)$ and $u=u(x, y, t)$. The invariant ω also has a y-dependence in this case.

Let us take

$$G=\partial^{-1}\omega_y+\gamma\omega,\quad \omega=u_x-u^2,$$
$$\partial^{-1}\omega_y=\int_0^x \omega_y\,d\xi, \tag{4.1}$$

where γ is a constant, then eq. (1.7) has the form

$$u_t+u_{xxy}-2u_x\partial^{-1}u_y^2-4u^2u_y+\gamma(u_{xxx}-6uu_x^2)$$
$$=0. \tag{4.2}$$

Substitution of eq. (4.1) into eq. (1.15) gives the equation

$$\omega_t+\omega_{xxy}+4\omega\omega_y+2\omega_x\partial^{-1}\omega_y+\gamma\omega_{xxx}+6\gamma\omega_x\omega$$
$$=0. \tag{4.3}$$

Equations (4.2) and (4.3) have been used to describe the interaction of a Riemann wave with a long wave [17]. It is obvious that eq. (4.2) is the modified equation of (4.3). These equations have Lax pairs and can be solved by means of the inverse scattering transform. They also have exact solutions, which are expressed in terms of the Weierstrass elliptic function.

Now let us show that the Kadomtsev–Petviashvili equation [18] can be obtained using an analogous approach.

Taking into account the truncated expansion (1.2) at $Z=Z(x, y, t)$ we have

$$\partial^{-1}u_y=-Z_yW+\partial^{-1}V_y. \tag{4.4}$$

We can write the following expression,

$$u_x-u^2-\partial^{-1}u_y=(-Z_{xx}+2vZ_x+Z_y)W$$
$$+v_x-v^2-\partial^{-1}v_y. \tag{4.5}$$

Assuming

$$v=\frac{Z_{xx}}{2Z_x}-\frac{Z_y}{2Z_x} \tag{4.6}$$

in eq. (4.5) we get the invariant

$$I=u_x-u^2-\partial^{-1}u_y=v_x-v^2-\partial^{-1}v_y, \tag{4.7}$$

if we take the transformation (4.6) and

$$u=-\frac{Z_x}{Z_0+Z}+\frac{Z_{xx}}{2Z_x}-\frac{Z_y}{2Z_x}. \tag{4.8}$$

Substituting (4.6) or (4.8) into eq. (4.7) we have

$$I=\tfrac{1}{2}\{Z;x\}-\frac{Z_y^2}{4Z_x^2}-\frac{\partial}{\partial x}(Z_y/Z_x)$$
$$+\tfrac{1}{2}\partial^{-1}(Z_y/Z_x)_y. \tag{4.9}$$

It is obvious that I is invariant under the Möbius group. From (1.2) we can also obtain the following equality,

$$(\partial/\partial y-\partial^{-1}u_y)u=(vZ_y-Z_{xy}+Z_x\partial^{-1}v_y)W$$
$$+(\partial/\partial y-\partial^{-1}v_y)v. \tag{4.10}$$

Taking into account eqs. (1.2), (4.7) and (4.10) we have the modified Kadomtsev–Petviashvili equation [19,20]

$$u_t-\frac{1}{4}\frac{\partial}{\partial x}[(\partial/\partial x+2u)I+4(\partial/\partial y-\partial^{-1}u_y)u]$$
$$+\frac{3}{4}\frac{\partial}{\partial y}I=0. \tag{4.11}$$

This equation can be obtained by considering the following equality,

$$u_t-\frac{1}{4}\frac{\partial}{\partial x}[(\partial/\partial x+2u)I+4(\partial/\partial y-\partial^{-1}u_y)u]+\tfrac{3}{4}I_y$$
$$=-\frac{\partial}{\partial x}[W(Z_t-\tfrac{1}{2}Z_xI+vZ_y-Z_{xy}+Z_x\partial^{-1}v_y)]$$
$$+v_t-\frac{1}{4}\frac{\partial}{\partial x}[(\partial/\partial x+2v)I+4(\partial/\partial y-\partial^{-1}v_y)v]$$
$$+\tfrac{3}{4}I_y. \tag{4.12}$$

Theorem 4.1. The condition of compatibility of the set of linear equations

$$Z_y = Z_{xx} - 2vZ_x \,, \qquad (4.13)$$

$$Z_t - \tfrac{1}{2}Z_x I + Z_y v + \partial^{-1} v_y Z_x - Z_{xy} = 0 \,, \qquad (4.14)$$

gives eq. (4.11).

Proof. This follows from the equality

$$(Z_y)_t = (Z_t)_y \,.$$

Theorem 4.2. Let $Z(x, y, t)$ be a solution of the equation

$$Z_t - \tfrac{1}{4}Z_x\{Z; x\} - \frac{3Z_y^2}{8Z_x} - \tfrac{3}{4}Z_x \partial^{-1}(Z_y/Z_x)_y = 0 \qquad (4.15)$$

then the expressions

$$u = -\frac{Z_x}{Z_0 + Z} + \frac{Z_{xx}}{2Z_x} - \frac{Z_y}{2Z_x} \qquad (4.16)$$

or

$$v = \frac{Z_{xx}}{2Z_x} - \frac{Z_y}{2Z_x} \qquad (4.17)$$

allow one to find the solutions of eq. (4.11).

Proof. This follows from the equality

$$u_t - \frac{1}{4}\frac{\partial}{\partial x}\left[(\partial/\partial x + 2u)I + 4(\partial/\partial y - \partial^{-1}u_y)u\right] + \tfrac{3}{4}I_y$$

$$= \left[\frac{\partial}{\partial x}\left(\frac{1}{2Z_x}\frac{\partial}{\partial x} - W\right) - \frac{1}{2Z_x}\frac{\partial}{\partial y} + \frac{Z_y}{2Z_x^2}\frac{\partial}{\partial x}\right]$$

$$\times (Z_t - \tfrac{1}{2}Z_x I + Z_y v + \partial^{-1} V_y Z_x - Z_{xy}) \,. \qquad (4.18)$$

By applying the Miura operator

$$\frac{\partial}{\partial x} - 2u - \partial^{-1}\frac{\partial}{\partial y} \qquad (4.19)$$

to eq. (4.11) we get the Kadomtsev–Petviashvili equation

$$I_t - \tfrac{1}{4}I_{xxx} - \tfrac{3}{2}II_x - \tfrac{3}{4}\partial^{-1}I_{yy} = 0 \,. \qquad (4.20)$$

One can look for the solutions of eq. (4.20) using eq. (4.9) and solutions of eq. (4.15). The Lax pair for eq. (4.20) can be obtained from eqs. (4.7) and (4.11).

5. Burgers equations in 2+1 dimension

We will construct these equations by using the truncated expansion in the form [21,22]

$$u = Z_x W, \quad W = \frac{1}{Z_0 + Z}, \quad Z = Z(x, y, t) \,. \qquad (5.1)$$

We have

$$Z_{k,x} W = (\partial/\partial x + u)^{k-1} u \quad (k = 1, \ldots, K) \,, \qquad (5.2)$$

$$Z_{n,y} W = (\partial/\partial x + \partial^{-1}u_y)^{m-1}\partial^{-1}u_y \quad (n = 1, \ldots, N) \,, \qquad (5.3)$$

$$Z_{m,y,k,x} W = (\partial/\partial x + \partial^{-1}u_y)^m (\partial/\partial x + u)^{k-1} u \quad (m = 0, \ldots, M) \,, \qquad (5.4)$$

where

$$Z_{k,x} = \frac{\partial^k Z}{\partial x^k}, \quad Z_{m,y,k,x} = \frac{\partial^{m+k} Z}{\partial y^m \partial x^k} \,. \qquad (5.5)$$

These formulas can be proved by the method of mathematical induction [7].

Taking into account equalities (5.2)–(5.4) we can write the family of equations

$$u_t = \frac{\partial}{\partial x}\left(\sum_{k=1}^{K} \alpha_k (\partial/\partial x + u)^{k-1} u\right) + \frac{\partial}{\partial x}\left(\sum_{n=1}^{N} \beta_n (\partial/\partial y + \partial^{-1}u_y)\partial^{-1}u_y\right) + \frac{\partial}{\partial x}\left(\sum_{m=0}^{M}\sum_{k=1}^{K} \gamma_{mk}(\partial/\partial x + \partial^{-1}u_y)^m \times (\partial/\partial x + u)^{k-1} u\right) , \qquad (5.6)$$

where α_k, β_n and γ_{mk} are coefficients of eqs. (5.6).

Substitution of eqs. (5.2), (5.3) and

$$u_t = -\frac{\partial}{\partial x}(Z_t W)$$

into (5.6) shows that the solutions of eq. (5.6) can be obtained by formula (5.1) if $Z(x, y, t)$ satisfies the equation

$$Z_t = \sum_{k=1}^{K} \alpha_k Z_{k,x} + \sum_{n=1}^{N} \beta_n Z_{n,y} + \sum_{m=0}^{M} \sum_{k=1}^{K} \gamma_{mk} Z_{m,y,k,x} \,. \tag{5.7}$$

It is clear that eqs. (5.6) have the Painlevé property and that they are integrable equations.

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