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Partial differential equations with solutions having movable first-order singularities

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This paper deals with the examination of nonlinear partial differential equations with solutions which have movable singularities of first order. Multi-phase and rational solutions of the family of nonintegrable equations are demonstrated. The equations can be integrable in other cases depending on the correlations of the coefficients of these equations.

1. Introduction

One of the impressive methods for the investigation of nonlinear partial differential equations is the application of the Painlevé test [1]. This approach has been applied both for integrable and some nonintegrable equations. As a rule it has been used for the investigation of concrete nonlinear partial differential equations. However, Weiss has considered the integrability of families of nonlinear partial differential equations having solutions with second-order singularities [2]. In this Letter we attempt to use the Painlevé property to construct the family of the partial differential equations having first-order singularities.

The outline of this Letter is as follows: in section 2 we describe the family of equations studied and prove an important lemma; in section 3 we discuss the forms of the solutions of the family of equations; in section 4 we give the solutions of the second-order partial differential equations which form our family of equations, and in section 5 we discuss our results.

2. The family of equations studied

Let us consider the following family of partial differential equations,

$$u_t=\sum_{n=0}^{N}\alpha_n L^n u+u\sum_{m=0}^{M}\beta_m L^m u\,, \tag{2.1}$$

where the operator L^n is defined by the following expression,

$$L^n u=(\partial/\partial x+u)^n u\,, \tag{2.2}$$

α_n and β_m are the coefficients of eqs. (2.1).

Note that the Newell–Whitehead [3], Burgers [4,5], generalized Fisher [6,7] equations and others are particular cases of (2.1).

In fact we have for $N=2$ and $M=1$ the generalized Fisher equation [6]

$$u_t=\alpha_0 u+(\alpha_1+\beta_0)u^2+(\alpha_2+\beta_1)u^3+\alpha_1 u_x+\alpha_2 u_{xx}+(3\alpha_2+\beta_1)uu_x\,. \tag{2.3}$$

We can obtain the Newell–Whitehead equation from eq. (2.3),

$$u_t=\alpha_0 u-2\alpha_2 u^3+\alpha_2 u_{xx}\,, \tag{2.4}$$

if $\beta_0=-\alpha_1=0$ and $\beta_1=-3\alpha_2$.

We get the Burgers equation from eq. (2.3),

$$u_t=2\alpha_2 uu_x+\alpha_2 u_{xx}\,, \tag{2.5}$$

in the case $\alpha_0=\alpha_1=\beta_0=\alpha_2+\beta_1=0$.

Assuming $N=3$ and $M=2$ in eq. (2.1) gives the generalized Korteweg–de Vries equation in the form

$$u_t=\alpha_0 u+(\alpha_1+\beta_0)u^2+(\alpha_2+\beta_1)u^3+(\alpha_3+\beta_2)u^4$$
$$+\alpha_1 u_x+\alpha_2 u_{xx}+\alpha_3 u_{xxx}+(3\alpha_2+\beta_2)uu_x$$
$$+(4\alpha_3+\beta_2)uu_{xx}+(3\beta_2+6\alpha_3)u^2u_x+3\alpha_3 u_x^2 . \tag{2.6}$$

We need the following statement.

Lemma 2.1. Let

$$u=z_x W(z) , \quad z=z(x,t) \tag{2.7}$$

and

$$\frac{dW}{dz}=-W^2 , \tag{2.8}$$

then

$$z_{m,x}W(z)=(\partial/\partial x+u)^{m-1}u \quad (m=1, ..., M) , \tag{2.9}$$

where $z_{m,x}$ is the mth order derivative with respect to x, $z(x, t)$ is a new function.

Proof. We apply the method of mathematical induction. In fact we have by definition

$$z_x W(z)=u=(\partial/\partial x+u)^0 u . \tag{2.10}$$

Then

$$z_{xx}W=(\partial/\partial x+u)u$$

after differentiation of eq. (2.10) with respect to x.

Let us suggest that $m=k$ in eq. (2.9), then

$$z_{k+1,x}W=\frac{\partial}{\partial x}[(\partial/\partial x+u)^{k-1}u]-z_{k,x}z_x\frac{dW}{dz} \tag{2.11}$$

and eq. (2.11) is transformed to the equation

$$z_{k+1,x}W=(\partial/\partial x+u)^k u , \tag{2.12}$$

taking into account eqs. (2.7) and (2.8).

This expression proves lemma 2.1.

We have

$$u_t=z_{xt}W-z_x z_t W^2 , \tag{2.13}$$

taking into account (2.7).

We obtain

$$u_t-\sum_{n=0}^{N}\alpha_n L^n u-u\sum_{m=0}^{M}\beta_m L^m u$$
$$=\left(z_{xt}-\sum_{n=0}^{N}\alpha_n z_{n+1,x}\right)W$$
$$-z_x\left(z_t+\sum_{m=0}^{M}\beta_m z_{m+1,x}\right)W^2 \tag{2.14}$$

after substitution of (2.9) and (2.13) into (2.1).

We will use expression (2.14) for the investigation of eqs. (2.1).

Remark 2.1. The solution of eq. (2.8) has the form

$$W(z)=\frac{1}{z_0+z} . \tag{2.15}$$

Therefore we obtain

$$u=\frac{z_x}{z_0+z} , \tag{2.16}$$

taking into account eqs. (2.7) and (2.15).

It follows from (2.16) now that $u(x, t)$ has movable singularities at $z_0+z(x,t)=0$.

3. Multi-phase and rational solutions of the family of equations (2.1)

Let us consider the solutions of eqs. (2.1) for $M=N-1$ (this value was chosen for the sake of convenience).

We will denote further

$$\gamma_k=\alpha_0 , \qquad k=0 ,$$
$$=\alpha_k+\beta_{k-1} , \quad k=1, ..., N . \tag{3.1}$$

We suppose in this Letter that the coefficients α_k and β_k in eq. (2.1) are constants. We hope to consider the case when α_k and β_k depend on x and t in another paper.

Definition 3.1. We will call the solutions (2.16) the N-phase solutions of eqs. (2.1), if $z(x, t)$ has the form

$$z(x,t)=C_0+\sum_{k=0}^{N}C_k\exp(\lambda_k x-\kappa_k t) , \tag{3.2}$$

where λ_k and κ_k are the constants to be determined, C_k $(k=0, ..., N)$ are arbitrary constants.

Theorem 3.1. If the values of the coefficients α_k and β_k in eqs. (2.1) are such that the equation

$$\sum_{k=0}^{N} \gamma_k \lambda^{k+1}=0 \tag{3.3}$$

has $N+1$ different roots λ_k $(k=0, ..., N)$, then eq. (2.1) has N-phase solutions in the form (2.16).

Proof. For eqs. (2.1) we have from (2.14)

$$W\left(z_{xt}-\sum_{n=0}^{N} \alpha_n z_{n+1,x}\right)$$

$$-z_x W^2\left(z_t+\sum_{m=0}^{N-1} \beta_m z_{m+1,x}\right)=0 . \tag{3.4}$$

Eq. (3.4) is satisfied if

$$z_{xt}-\sum_{n=0}^{N} \alpha_n z_{n+1,x}=0 \tag{3.5}$$

and

$$z_t+\sum_{m=0}^{N-1} \beta_m z_{m+1,x}=0 . \tag{3.6}$$

One obtains

$$\sum_{k=0}^{N} \gamma_k z_{k+1,x}=0 \tag{3.7}$$

from eqs. (3.5) and (3.6), where γ_k are the coefficients (3.1).

It is known that eq. (3.7) has $N+1$ independent solutions if the polynomial equation (3.3) has $N+1$ different roots. Let us denote these roots as λ_k: $\lambda_0=0$, $\lambda_k\neq 0$ $(k=1, ..., N)$, then the general solution of eq. (3.7) is

$$z(x, t)=\psi_0(t)+\sum_{k=1}^{N} \psi_k(t) \exp(\lambda_k x) . \tag{3.8}$$

Substitution of (3.8) into (3.6) gives the following equations for $\psi_k(t)$ $(k=0, ..., N)$,

$$\dot{\psi}_k+\psi_k \sum_{m=0}^{N} \beta_m(\lambda_k)^{m+1} \quad (k=0, ..., N) . \tag{3.9}$$

One finds

$$\psi_k(t)=C_k \exp(-\kappa_k t) , \tag{3.10}$$

$$\kappa_k=\sum_{m=0}^{N} \beta_m(\lambda_k)^{m+1} \quad (k=0, ..., N) \tag{3.11}$$

from eq. (3.9).

The dependences (3.8) and (3.10) prove theorem 3.1.

Definition 3.2. Solutions (2.16) will be called rational solutions of the family of equations (2.1) if $z(x, t)$ in (2.16) has the form of an Nth power algebraic polynomial in the variables x and t having $N+1$ constants.

Theorem 3.2. If the values of the coefficients α_k and β_k in eqs. (2.1) are such that $\gamma_k=0$ $(k=0, ..., N-1)$, but $\gamma_N\neq 0$, then eqs. (2.1) have the rational solutions (2.16).

Proof. One can obtain the equation

$$z_{N+1,x}=0 , \tag{3.12}$$

taking into account the conditions of the theorem and eq. (3.7).

The general solution of this equation has the form

$$z(x, t)=\varphi_0(t)+x\varphi_1(t)+...+x^N\varphi_N(t) . \tag{3.13}$$

Substitution of (3.13) in (3.6) gives the following equation for $\varphi_n(t)$ $(n=0, ..., N)$,

$$\dot{\varphi}_n+\sum_{k=0}^{N-n-1} \beta_k(k+n+1)!\varphi_{k+n+1} \quad (n=0, ..., N-1) ,$$

$$\varphi_N=C_N , \tag{3.14}$$

where C_N is an arbitrary constant.

The dependences $\varphi_n(t)$ in the from of polynomials in t can be found from (3.14).

Definition 3.3. The solutions (2.14) will be called combined solutions of the family of equations (2.1) if $z(x, t)$ is a polynomial in $C_k \exp(\lambda_k x-\kappa_k t)$ and $C_m(\lambda_m x-\kappa_m t)$.

Theorem 3.3. If the values of the coefficients α_k and β_k in the family of equations (2.1) are such that eq. (3.3) has multiple roots then eqs. (2.1) have combined solutions.

Proof. We arrive at this statement by analyzing the solutions of the ordinary differential equations (3.7) and solving eq. (3.6) under the conditions of this theorem.

Definition 3.4. Equations (2.1) will be called integrable equations, if their solutions (2.16) are expressed as solutions of the linear partial differential equations (3.6).

Theorem 3.4. If the values of coefficients α_k and β_k in eqs. (2.1) are such that $\gamma_k=0$ ($k=0, ..., N$) then eqs. (2.1) will be integrable equations.

Proof. Let $\alpha_0=0$ and the coefficients α_k and β_k in eqs. (2.1) satisfy the conditions $\alpha_k+\beta_{k-1}=0$ ($k=1, ..., N$) then eq. (3.5) is a consequence of eq. (3.6). In this case the solutions of eqs. (2.1) are defined by formula (2.16), where $z(x, t)$ satisfies a linear partial differential equation in the form

$$z_t=\sum_{n=1}^{N}\alpha_n z_{n,x}\,. \tag{3.15}$$

Remark 3.1. The integrable equations of the type (2.1) can be written in the form

$$u_t=\frac{\partial}{\partial x}\left(\sum_{n=1}^{N}\alpha_n(\partial/\partial x+u)^{n-1}u\right). \tag{3.16}$$

The solutions of eqs. (3.16) can be obtained from formula (2.16) if $z(x, t)$ satisfies linear equations of the type (3.15).

The next integrable equations of family (2.1) one gets for $N=2, 3, 4$:

$$u_t=\alpha_1 u_x+\alpha_2 u_{xx}+2\alpha_2 uu_x\,, \tag{3.17}$$

$$u_t=\alpha_1 u_x+\alpha_2 u_{xx}+\alpha_3 u_{xxx}+2\alpha_2 uu_x+3\alpha_3 uu_{xx}+3\alpha_3 u^2u_x+3\alpha_3 u_x^2\,, \tag{3.18}$$

$$u_t=\alpha_1 u_x+\alpha_2 u_{xx}+\alpha_3 u_{xxx}+\alpha_4 u_{xxxx}+2\alpha_2 uu_x+3\alpha_3 uu_{xx}+3\alpha_3 u^2u_x+3\alpha_3 u_x^2+4\alpha_4 u^3u_x+6\alpha_4 u^2u_{xx}+4\alpha_4 uu_{xxx}+12\alpha_4 uu_x^2+10\alpha_4 u_xu_{xxx}\,, \tag{3.19}$$

where the coefficients α_k are constants.

Equation (3.16) is the Burgers equation for $\alpha_1=0$. It is well known that the solutions of this equation are defined after solving the linear heat equation by means of the Cole–Hopf transformation [4,5]. The solution of eqs. (3.18) and (3.19) can be obtained using expression (2.16) and the solutions of the following linear partial differential equations,

$$z_t=\alpha_1 z_x+\alpha_2 z_{xx}+\alpha_3 z_{xxx}\,, \tag{3.20}$$

$$z_t=\alpha_1 z_x+\alpha_2 z_{xx}+\alpha_3 z_{xxx}+\alpha_4 z_{xxxx}\,. \tag{3.21}$$

Remark 3.2. The compatibility conditions of the two linear equations (3.5) and (3.6) are equivalent to the statement, that eqs. (2.1) have the Painlevé property [1]. Equations (2.1) have the conditional Painlevé property if $\alpha_k+\beta_{k-1}\neq 0$ ($k=1, ..., N$).

Remark 3.3. One can obtain the more general integrable and nonintegrable equations than the family (2.1) taking $u=g(v, x, t)$ in (2.1), where the function $g(v, x, t)$ is such that $g_v\neq 0$.

These equations have the form

$$g_v v_t=\frac{\partial}{\partial x}\sum_{n=0}^{N}\alpha_n(\partial/\partial x+g)^n g+E(v, v_x, ..., x, t)\sum_{m=0}^{M}\beta_m(\partial/\partial x+g)^m g\,, \tag{3.22}$$

where $E(v, v_x, ..., x, t)$ is a smooth function.

One can obtain the solutions v of eqs. (3.22) from the expression

$$g(v, x, t)=\frac{z_x}{z_0+z}\,, \tag{3.23}$$

if $z(x, t)$ satisfies the set of equations

$$z_t=\sum_{n=0}^{N}\alpha_n z_{n+1,x}\,, \tag{3.24}$$

$$\sum_{m=0}^{M}\beta_m z_{m+1,x}=0\,. \tag{3.25}$$

We note the result.

Theorem 3.5. The nonintegrable generalized Nth order equations of the type (2.1) have no solutions in the form (2.16) with k phases, for $k>N$.

Proof. The generalized Nth order equations of the type (2.1) can be obtained from eqs. (3.22). Equations (3.22) have M-phase solutions (2.16) because $z(x, t)$ is the solution of eqs. (3.24) and (3.25). However, for the Nth order equation we have $M \leqslant N$. All generalized nonintegrable Nth order equations of the type (2.1) can have exact solutions (2.16) with $k \leqslant M$ phases. This proves theorem 3.5.

4. The exact solutions of eqs. (2.1) for $N=2$, $M=1$

Let us obtain the solutions of eqs. (2.1) for $N=2$, $M=1$, because this case is one of the simplest.

Equation (3.3) turns into the following equation,

$$\alpha_0\lambda + (\alpha_1 + \beta_0)\lambda^2 + (\alpha_2 + \beta_1)\lambda^3 = 0 . \quad (4.1)$$

There are three roots of (4.1),

$$\lambda_0 = 0 ,$$

$$\lambda_{1,2} = -\frac{\beta_0 + \alpha_1}{2(\beta_1 + \alpha_2)} \pm \sqrt{\frac{4\alpha_0(\beta_1 + \alpha_2) - (\beta_0 + \alpha_1)^2}{4(\beta_1 + \alpha_2)^2}} . \quad (4.2)$$

for $\alpha_1 + \beta_2 \neq 0$.

There are six cases to be considered.

Case 1. $(\beta_0 + \alpha_1)^2 > 4\alpha_0(\beta_1 + \alpha_2)$. We get that eq. (2.3) has the two-phase solution (2.16) where $z(x, t)$ is

$$z(x, t) = C_0 + C_1 \exp[\lambda_1 x - \lambda_1(\beta_0 + \beta_1\lambda_1)t] + C_2 \exp[\lambda_2 x - \lambda_2(\beta_0 + \beta_1\lambda_2)t] . \quad (4.3)$$

Case 2. $\alpha_0 = 0$, $\beta_0 = -\alpha_1$. We find from (4.2) that

$$\lambda_1 = \lambda_2 = \lambda_3 = 0 . \quad (4.4)$$

The solution $z(x, t)$ of eqs. (3.6), (3.7) takes the form

$$z(x, t) = C_0 + C_1[x - (\beta_0 + 2\beta_1 C_2)t] + C_2(x - \beta_0 t)^2 . \quad (4.5)$$

Case 3. $(\beta_0 + \alpha_1)^2 < 4\alpha_0(\beta_1 + \alpha_2)$. Equation (3.7) has the solution

$$z(x, t) = C_0 + C_1 \exp\{-a_0 x - [a_0\beta_0 + (a_0^2 + b_0^2)\beta_1]t\} \times \sin[b_0 x - b_0(\beta_0 - 2\alpha_0\beta_1)t + C_2] \quad (4.6)$$

in this case, where

$$a_0 = \frac{\beta_0 + \alpha_1}{2(\beta_1 + \alpha_2)} ,$$

$$b_0 = \sqrt{\frac{4\alpha_0(\beta_1 + \alpha_2) - (\beta_0 + \alpha_1)^2}{4(\beta_1 + \alpha_2)^2}} .$$

Case 4. $(\beta_0 + \alpha_1)^2 = 4\alpha_0(\beta_1 + \alpha_2)$. We have from eq. (4.2)

$$\lambda_0 = 0 , \quad \lambda_1 = \lambda_2 = -\frac{\beta_0 + \alpha_1}{2(\beta_1 + \alpha_2)} . \quad (4.7)$$

The solution of eqs. (3.6), (3.7) is

$$z(x, t) = C_0 + \{C_1 + C_2[x - (\beta_0 + 2\beta_1\lambda_1)t]\} \times \exp[\lambda_1 x - \lambda_1(\beta_0 + \beta_1\lambda_1)t] . \quad (4.8)$$

Case 5. $\alpha_0 = 0$. We find from eq. (4.2)

$$\lambda_0 = \lambda_1 = 0 , \quad \lambda_2 = -\frac{\beta_0 + \alpha_1}{\beta_1 + \alpha_2} . \quad (4.9)$$

The solution of eqs. (3.6), (3.7) takes the form

$$z(x, t) = C_0 + C_1(x - \beta_0 t) + C_2 \exp[\lambda_1 x - \lambda_1(\beta_0 + \beta_1\lambda_1)t] . \quad (4.10)$$

Substitution of (4.3), (4.5), (4.6), (4.8), and (4.10) into (2.16) gives two-phase rational and combined solutions of eq. (2.3).

Case 6. $\alpha_0 = \alpha_1 + \beta_0 = \alpha_2 + \beta_1 = 0$. Equation (2.3) is integrable in this case and its solutions are found by formula (2.16), where $z(x, t)$ is one of the linear partial differential equations of second order

$$z_t = \alpha_1 z_x + \alpha_2 z_{xx} . \quad (4.11)$$

We can consider by analogy the solutions of the family of equations (2.1) in other cases.

5. Discussion

The family of equations which have exact solutions with a singularity of first order was studied in this paper. This family consists of the Burgers, Newell–Whitehead, generalized Fisher equations, and others. All these equations have solutions of one kind. For some values of the coefficients of the family of equations (2.1) these equations become integrable. It has turned out that the nonintegrable equations of Nth order have N-phase, rational and combined solutions with $N+1$ arbitrary constants. The nonlinear second-order equations of the type (2.1) were examined in detail to illustrate this fact. Among the interesting results obtained in this work are the following. Nonintegrable Nth order equations can have solutions with $N+1$ arbitrary constants and these solutions have a direct connection with the solutions of the ordinary differential equation of Nth order.

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