On types of nonlinear nonintegrable equations with exact solutions

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Some nonlinear nonintegrable equations of evolution type have been investigated. Solutions of nonlinear equations frequently used in various fields of physics are expressed in terms of the solutions of the Riccati equation and the equation for the anharmonic oscillator.

1. Introduction

Solutions of some group of nonlinear partial differential equations for two independent variables,

$$E(x, t, u, u_t, u_x, ...)=0\,, \tag{1.1}$$

can be represented in terms of the Painlevé expansions [1]

$$u=\phi(x, t)^p \sum_{j=0}^{\infty} u_j(x, t)\phi^j\,, \tag{1.2}$$

where p is an integer, $\phi(x, t)$ and u_j are analytic functions in a neighborhood of the manifold $\phi(x, t)=0$.

As a rule, the expansion of the solution of eq. (1.1) in terms of (1.2) simplifies the procedure of deriving analytic solutions of the equations under study. The expansion (1.2) allows one to find the Lax pair and Bäcklund transformations [1] in the equations integrable by means of the inverse scattering transform. The same expansion can bring about some exact solutions [2,3] in the case of nonintegrable original equations.

Let us consider an application of the truncated expansion (1.2) where $u_j=0$ at $j>p$, i.e., when the solution of the nonlinear equations can be written as

$$u=\sum_{j=0}^{p} u_j\phi^{j-p}\,. \tag{1.3}$$

Substituting (1.3) into (1.1) and equating the expressions of the same powers of $\phi(x, t)$ to zero, one can find the dependences $u_j(x, t)$ on the derivatives of $\phi(x, t)$, relation (1.3) being represented as

$$u=u_p+\sum_{i=0}^{1} A_i \frac{\partial \ln \phi}{\partial x^i \partial t^{1-i}}+\sum_{k=0}^{2} A_{k,2-k} \frac{\partial^2 \ln \phi}{\partial x^k \partial t^{2-k}}+\dots + \sum_{l=0}^{p} A_{l,p-l} \frac{\partial^p \ln \phi}{\partial x^l \partial t^{p-l}}\,. \tag{1.4}$$

Here A_i, $A_{k,2-k}$, ..., are constants.

The expansions of the solutions for the nonlinear equations (1.4) are now well-known. The Cole–Hopf transformations for the Burgers equations and other transformations for solutions of (1.4) were efficiently used by Hirota and other investigators to find exact solutions for equations integrable by means of the inverse scattering transform [1,4]. However, if the dependence of u_p on the derivatives of $\phi(x, t)$ is taken into account in (1.4), then the latter can be reduced to

$$u=F(C, S, C_x, S_x, ...)+\sum_{i=0}^{1} A_i \frac{\partial \ln(\phi/\sqrt{\phi_x})}{\partial x^i \partial t^{1-i}} + \sum_{k=0}^{2} A_{k,2-k} \frac{\partial^2 \ln(\phi/\sqrt{\phi_x})}{\partial x^k \partial t^{2-k}}+\dots + \sum_{l=0}^{p} A_{l,p-l} \frac{\partial^p \ln(\phi/\sqrt{\phi_x})}{\partial x^l \partial t^{p-l}}\,. \tag{1.5}$$

Here

$$C=-\frac{\phi_t}{\phi_x}, \quad S=\{\phi; x\}=\frac{\phi_{xxx}}{\phi_x}-\frac{3}{2}\frac{\phi_{xx}^2}{\phi_x^2}, \tag{1.6}$$

and S is the Schwarzian derivative.

The statement above can be rigorously proved but it is rather cumbersome for the general class of equations; we shall illustrate our idea by expanding some concrete equations as examples.

2. Examples of Painlevé expansions for nonlinear nonintegrable equations

Let us consider expansions of the solutions of the type (1.4) and (1.5) for some equations.

(1) The Burgers–Korteweg–de Vries equation is often used in the description of wave processes in dissipative–dispersive systems in many areas or physics. It has the form

$$u_t+uu_x+\beta u_{xxx}=\nu u_{xx}. \tag{2.1}$$

When the solution is sought in terms of (1.3) then $u(x, t)$ can be represented as [5]

$$u=u_2-\tfrac{12}{5}\nu\frac{\partial \ln\phi}{\partial x}+12\beta\frac{\partial^2 \ln\phi}{\partial x^2}. \tag{2.2}$$

Taking into account the dependence of u_2 on the derivatives of $\phi(x, t)$,

$$u_2=C+2\beta S+\tfrac{1}{25}\nu^2/\beta+\tfrac{6}{5}\nu\frac{\partial \ln\phi_x}{\partial x}-6\beta\frac{\partial^2 \ln\phi_x}{\partial x^2}, \tag{2.3}$$

one can write expression (2.2) in the form

$$u=C+2\beta S+\tfrac{1}{25}\nu^2/\beta-\tfrac{12}{5}\nu Y+12\beta Y_x, \tag{2.4}$$

where

$$C=-\frac{\phi_t}{\phi_{xx}}, \quad S=\{\phi; x\}=\frac{\phi_{xxx}}{\phi_x}-\frac{3}{2}\frac{\phi_{xx}^2}{\phi_x^2}, \tag{2.5}$$

$$Y=\frac{\partial}{\partial x}\ln\frac{\phi}{\sqrt{\phi_x}}, \tag{2.6}$$

and S is the Schwarzian derivative.

(2) The generalized Kuramoto–Sivashinsky equation is one of the main equations in the physics of unstable systems [5,6],

$$u_t+\alpha_0 u^r u_x+u_{xx}+\sigma u_{xxx}+u_{xxxx}=0. \tag{2.7}$$

For $\alpha_0=r=1$, it has a solution transform [7]

$$u=u_3+\tfrac{15}{76}(16-\sigma^2)\frac{\partial \ln\phi}{\partial x}+15\sigma\frac{\partial^2 \ln\phi}{\partial x^2}$$
$$+60\frac{\partial^3 \ln\phi}{\partial x^3} \tag{2.8}$$

and bearing in mind the dependence of u_p on the derivatives of $\phi(x, t)$ from ref. [8],

$$u=C+\tfrac{5}{2}\sigma S+15S_x+\tfrac{1}{76}\sigma(7-\tfrac{13}{8}\sigma)^2+\tfrac{15}{76}(16-\sigma^2)Y$$
$$+15\sigma Y_x+60Y_{xx}. \tag{2.9}$$

For the case when $\alpha_0=24$ (this value was chosen for the sake of convenient calculations) the solutions of the generalized Kuramoto–Sivashinsky equation (2.7) can be written in the form

$$u=u_1+\frac{\partial \ln\phi}{\partial x}, \quad u=-\tfrac{1}{12}\sigma+Y. \tag{2.10}$$

(3) The Kawachara equation describing nonlinear wave processes in the dispersive system [9,10]

$$u_t+uu_x+u_{xxx}=u_{xxxxx} \tag{2.11}$$

has an expansion (1.4) and (1.5) at $p=4$ of the type

$$u=u_4+\frac{280}{13}\frac{\partial^2 \ln\phi}{\partial x^2}-280\frac{\partial^4 \ln\phi}{\partial x^4},$$

$$u=C+\tfrac{140}{39}S-\tfrac{28}{3}S^2-84S_x-\tfrac{31}{507}+\tfrac{280}{13}Y_x$$
$$-280Y_{xxx}. \tag{2.12}$$

(4) The Bretherton equation [11]

$$u_{tt}+u_{xx}+u_{xxxx}+u=\alpha_0 u^3, \tag{2.13}$$

for $\alpha_0=30$ (chosen for convenience), has an expansion (1.4) at $p=2$ in the form

$$u=u_2+2\frac{\partial^2 \ln\phi}{\partial x^2}. \tag{2.14}$$

After substituting the dependence of u_2 on the derivatives of $\phi(x, t)$ into (2.14) one has

$$u=-\tfrac{1}{30}(C^2+1)+\tfrac{2}{3}SY_x. \tag{2.15}$$

(5) The equation of thermal conductivity with a nonlinear source,

$$u_t=u_{xx}+a_0+a_1u+a_2u^2-2u^3, \tag{2.16}$$

has the following expansion solutions (1.4) and (1.5),

$$u=u_1-\frac{\partial \ln \phi}{\partial x}, \qquad (2.17)$$

$$u=\tfrac{1}{6}(C+a_2)-Y. \qquad (2.18)$$

The transformations given in this section can be used to derive exact solutions of some nonlinear equations.

3. The properties of the Painlevé expansions

We shall consider in more detail the properties of expansions of eqs. (1.4), (1.5) which are to be accounted for. These properties can be formulated as a number of theorems.

Theorem 1. Let $S=\{\phi, x\}$ be the Schwarzian derivative, then the function Y defined by eq. (2.6) must satisfy the Riccati equation in the form

$$Y_x=-Y^2-\tfrac{1}{2}S. \qquad (3.1)$$

The theorem can be proved by direct substitution of expression (2.5), for S, and (2.6), for Y, into (3.1).

Corollary 1. The mth order derivatives ($m=2$, 3, ...) of Y with respect to x are expressed in terms of a polynomial of the ($m+1$)st order with coefficients depending on S and its derivatives.

Indeed, differentiating (3.1) with respect to x, one has

$$Y_{xx}=2Y^3+SY-\tfrac{1}{2}S_x, \qquad (3.2)$$

$$Y_{xxx}=-6Y^4-4SY^2+S_xY-\tfrac{1}{2}S^2-\tfrac{1}{2}S_{xx},$$

$$\ldots,$$

$$Y_{mx}=(-1)^m m!\,Y^{m+1}-\ldots-\tfrac{1}{2}S_{m-1,x}, \qquad (3.3)$$

where Y_{mx}, S_{mx} are the mth order derivatives of Y and S with respect to x.

Theorem 2. Let $S=\{\phi, x\}$ be the Schwarzian derivative and $C=-\phi_t/\phi_x$, then the function Y defined by expression (2.6) satisfies the Riccati equation in the form

$$Y_t=CY^2-C_xY+\tfrac{1}{2}(CS+C_{xx}). \qquad (3.4)$$

The proof of the theorem can be demonstrated by substituting expressions (2.5) and (2.6) for Y, C and S into (3.4).

Corollary 2. The kth order derivatives of Y with respect to x and t are expressed by means of a ($m+1$)st order polynomial of Y with factors depending on C, S, and their derivatives.

By differentiating eq. (3.4) m times with respect to t and x, and taking into account (3.2) and (3.4), one shows the validity of the statement.

Corollary 3. Expansion of the solutions (1.5) can be presented as a number of polynomials of Y with factors depending on the functions S, C, and their derivatives, the power of the polynomial being p.

This statement follows from the expansion (1.5) after substituting expressions (3.2)–(3.4) into it.

Theorem 3. The Schwarzian derivative S and the function C are invariant with respect to the Möbius transformation group

$$\phi=\frac{a\pm b\psi}{c+d\psi}, \quad ad-bc=1. \qquad (3.5)$$

Proof. Substituting (3.5) into (2.6) for S and C one finds [12]

$$\{\phi, x\}=\{\psi, x\}, \quad \phi_t/\phi_x=\psi_t/\psi_x.$$

Theorem 4. Let S and C be described by expressions (2.5) and (2.6), then the requirement of compatibility for S and C is satisfied by the equation

$$S_t+C_{xxx}+2C_xS+CS_x=0. \qquad (3.6)$$

Proof. Remembering that

$$(\phi_t)_{xxx}=(\phi_{xxx})_t$$

and the dependences (2.5) and (2.6) for C and S on the variables $\phi(x, t)$ we have eq. (3.6) [13].

4. The algorithm of finding exact solutions of the nonlinear equations

The expansion of the solutions for eq. (1.1) in the form of (1.5) can result in exact solutions of nonlinear equations. Let us assume further in (1.5)

$$\frac{\partial}{\partial x}\ln\frac{\phi}{\sqrt{\phi_x}}=Y_1\,,\quad \frac{\partial}{\partial t}\ln\frac{\phi}{\sqrt{\phi_x}}=Y_2\,,\quad \ldots,$$

and substitute the transformation into the initial equation, taking into account the relations of the type (3.1)–(3.4). Setting the equations with the same powers of Y_1, Y_2, Y_1Y_2, ..., equal to zero, we arrive at a set of nonlinear differential equations with respect to $C(x,t)$ and $S(x,t)$. Integrating this system on account of eq. (3.6) and finding $Y(x,t)$ from eqs. (3.1)–(3.4) on account of (1.5), one can obtain the exact solution of the original equation (1.1). However, our observations showed that the system of differential equations with respect to $C(x,t)$ and $S(x,t)$ thus obtained is not simpler for study, as a rule, than the original equation is. In addition, the solution of the set of equations with respect to $C(x,t)$ and $S(x,t)$ is possible in the same cases if and only if C and S are constants.

The algorithm becomes simpler if one can find the exact solutions of the initial equations expressed in terms of a travelling wave variable,

$$u(x,t)=U(\xi)\,,\quad \xi=x-C_0t\,, \tag{4.1}$$

where C_0 is the wave velocity. In this case

$$C=-\phi_t/\phi_x=C_0\,.$$

It follows from (3.6) that for $C_0\neq 0$, S is also a constant. It should be noted that in ordinary differential equations, when $C=0$, S can be a function of x. Eq. (3.4), when $\xi=x-C_0t$, reduces to (3.1) with $x\to\xi$,

$$Y_\xi=-Y^2-\tfrac{1}{2}S\,. \tag{4.2}$$

Eq. (1.5) coincides with eq. (1.4) at constant C and S if, in (1.4), $\phi\to\phi/\sqrt{\phi_\xi}$, and it is assumed that $u_p=\text{const}$. Therefore the solution of the nonlinear equation (1.1) in terms of a travelling wave variable for which there exists an expansion of the type (1.2) can be sought in the form

$$U=C_1+C_2Y+\ldots+C_{p+1}Y_{p-1,\xi}\,,\quad Y_{k,\xi}=\mathrm{d}Y^k/\mathrm{d}\xi^k\,, \tag{4.3}$$

where the C_k are constants. A function $Y(\xi)$ can be obtained from eqs. (3.1)–(3.3). If $A_i=0$ ($i=0,1$) in (1.5) and $C_2=0$ in (4.3), respectively, then the solution of the investigated equation can be found in terms of the travelling wave variable as

$$U=C_1+C_3R+\ldots+C_{p+1}R_{p-2,\xi}\,,\quad R_\xi=Y_{1,\xi}\,, \tag{4.4}$$

where $R(\xi)$ can satisfy the equation for the anharmonic oscillator as follows from eq. (3.3) at $S=\text{const}$,

$$(R_\xi)^2=-4R^3-2S_0R^2+2bR+d\,. \tag{4.5}$$

The constants S in eq. (4.2) and S_0, b and d in eq. (4.5) are found by substituting (4.3) or (4.4) into eq. (1.1).

Eqs. (3.1)–(3.3) and hence eqs. (4.2) and (4.5) have no movable critical points and are in fact of Painlevé type [1]. Therefore we will try to solve some nonintegrable differential equations (which do not satisfy the Painlevé property) by means of solutions of the Painlevé type equations with a smaller order. We also plan a further investigation of this idea and its possible application for nonlinear equations.

5. Exact solutions of some nonlinear nonintegrable equations

Let us show how the suggested algorithm works on the examples of the equations given in section 2.

(1) Substituting the solution expansion (2.4) for the Burgers–Korteweg–de Vries equation written in terms of a travelling wave variable, one finds

$$\beta U_{\xi\xi}-\nu U_\xi+\tfrac{1}{2}U^2-c_0U+q=12\beta^2E_{\xi\xi}$$
$$-24\beta(\tfrac{3}{5}\nu-\beta Y)E_\xi+12(\tfrac{7}{25}\nu^2-4\beta^2Y)E+48\beta^2E^2\,,$$
$$\xi=x-c_0t\,,\quad c_0^2=2q+\tfrac{36}{625}\nu^4/\beta^2\,, \tag{5.1}$$

where q is the integration constant of the Burgers–Korteweg–de Vries equation (2.1) which appears after transforming into the variables (4.1). The function E has the form

$$E=Y_\xi+Y^2+\tfrac{1}{2}S\,,\quad S=-\tfrac{1}{50}\nu^2/\beta^2\,. \tag{5.2}$$

It follows from (5.1) that if $E=0$, i.e., $Y(\xi)$ is the solution of the Riccati equation (4.2),

$$Y(\xi)=\kappa\frac{A_0 e^{\kappa\xi}-B_0 e^{-\kappa\xi}}{A_0 e^{\kappa\xi}+B_0 e^{-\kappa\xi}}=\kappa\tanh(\kappa\xi+\varphi_0)\,,$$

$$\kappa^2=-\tfrac{1}{2}S=\tfrac{1}{50}\nu^2/\beta^2\,, \qquad (5.3)$$

where A_0, B_0 and φ_0 are arbitrary constants, then $U(\xi)$ defined by eq. (2.4) is a kink solution of the Burgers–Korteweg–de Vries equation,

$$U(\xi)=C_0-\tfrac{12}{5}\nu\kappa\tanh(\kappa\xi+\varphi_0)$$
$$+12\beta\kappa^2\,\mathrm{ch}^{-2}(\kappa\xi+\varphi_0)\,,$$
$$\kappa=\pm\tfrac{1}{10}\nu/\beta\,. \qquad (5.4)$$

(2) Substituting the transformation (2.15) to solve the Bretherton equation in the form

$$U=-\tfrac{1}{30}(C_0^2+1)+\tfrac{1}{3}S_0+2R(\xi) \qquad (5.5)$$

for eq. (2.13) written in terms of the travelling wave variable, one finds

$$(1+C_0^2)U_{\xi\xi}+U_{\xi\xi\xi\xi}+U-30U^3$$
$$=(24R+4S_0-2C_0^2-2)Z+48\int R_\xi Z\,\mathrm{d}\xi\,, \qquad (5.6)$$

where

$$Z=R_{\xi\xi}+6R^2+2S_0R-b\,,$$
$$b=-\tfrac{1}{360}(60S_0^2+C_0^4+2C_0^2-9)\,. \qquad (5.7)$$

It is seen from (5.6) that if $Z=0$, then the solution for the anharmonic oscillator,

$$(R_\xi)^2+4R^3+2S_0R^2-2bR-d=0\,, \qquad (5.8)$$

where

$$d=\tfrac{1}{5400}(4+2C_0^2-3C_0^4-C_0^6+45S_0-10S_0C_0^2$$
$$-5S_0C_0^4-100S_0^3)\,,$$

will be transformed according to (5.5) into the solution of the Bretherton equation.

Let R_1, R_2 and R_3 be the real roots of the cubic equation

$$R^3+\tfrac{1}{2}S_0R^2-\tfrac{1}{2}bR-\tfrac{1}{4}d=0\,, \qquad (5.9)$$

then the solution of (5.8) at $R_1\geqslant R_2\geqslant R_3$ is expressed by means of an elliptic Jacobi function as

$$R(\xi)=R_2+(R_1-R_2)\,\mathrm{cn}^2[\xi\sqrt{R_1-R_2},m]\,,$$
$$m^2=\frac{R_1-R_2}{R_1-R_3}\,. \qquad (5.10)$$

The solution of the Bretherton equation (2.13) can be represented in the form

$$U(\xi)=\tfrac{1}{30}(10S_0-C_0^2-1)-2R_2$$
$$-2(R_1-R_2)\,\mathrm{cn}^2[\xi\sqrt{R_1-R_2},m]\,. \qquad (5.11)$$

The analysis of the roots of the cubic equation (5.9) shows that the solution (5.11) of eq. (2.13) for $|C_0|\leqslant\sqrt{\tfrac{3}{2}}$ and S an arbitrary constant has a physical meaning and is a periodic (cnoidal) wave.

(3) Substituting the solution transformation (2.9) for the generalized Kuramoto–Sivashinsky equation for

$$\alpha_0=r=1\,,\quad \sigma=4\,,\quad S=S_0=\text{const}$$

into eq. (2.7) in terms of the travelling wave variable, one obtains

$$U_{\xi\xi\xi}+4U_{\xi\xi}+U_\xi+\tfrac{1}{2}U^2-C_0U+q$$
$$=60\Big(Z_{\xi\xi}+5Z_\xi+(5-12R-2S_0)Z$$
$$-36\int R_\xi Z\,\mathrm{d}\xi\Big)\,, \qquad (5.12)$$

where

$$Z=R_{\xi\xi}+6R^2+2S_0R-b\,,$$
$$b=\tfrac{1}{24}(1-4S_0^2)\,. \qquad (5.13)$$

It follows from (5.13) that $R(\xi)$ is a solution of the equation for the anharmonic oscillator (5.8), where

$$d=\tfrac{1}{2160}(C_0^2+30S_0-40S_0^3-2q-26)\,.$$

The expression

$$U(\xi)=C_0+10S_0-1+60R+60R_\xi \qquad (5.14)$$

gives the solution of eq. (2.7) for

$$\alpha_0=r=1\,,\quad \sigma=4\,,$$
$$q^2-q(C_0^2-26)+\tfrac{1}{4}C_0^4-\tfrac{13}{2}C_0^2+144\leqslant 0$$

and for an arbitrary constant S_0. All these solutions are periodic waves. The solution (5.8) transforms into a solitary wave for $q=d=0$, $C_0=6$, $S_0=-\tfrac{1}{2}$ ($R_1=\tfrac{1}{4}$, $R_2=R_3=0$), $R(\xi)=\tfrac{1}{4}\,\mathrm{ch}^{-2}(\tfrac{1}{2}\xi)$, which after substitution into (5.14) gives the solitary wave of eq. (2.7) [8],

$$U(\xi)=15\,\mathrm{ch}^{-2}(\tfrac{1}{2}\xi)[1-\tanh(\tfrac{1}{2}\xi)]\,.$$

(4) If the solution transformation (2.12) is substituted into the Kawachara equation (2.11) written in the travelling wave form,

$$U_{\xi\xi\xi\xi} - U_{\xi\xi} - \tfrac{1}{2}U^2 + C_0 U - q = 0\,, \qquad (5.15)$$

then by analogy with the previous cases one finds that (5.15) has the solution

$$U(\xi) = C_1 + \tfrac{280}{13}R - 280R_{\xi\xi}\,,$$
$$C_1 = C_0 - \tfrac{31}{507} - \tfrac{28}{3}S_0^2 - \tfrac{140}{39}S_0\,, \qquad (5.16)$$

where $R(\xi)$ also satisfies the equation for the anharmonic oscillator (5.8) with

$$b=0\,,\quad d=\tfrac{1}{27}S_0^3 - \tfrac{7}{2340}S_0^2 + \tfrac{31}{4775520}\,,$$
$$C_0^2 = 2q + 1288S_0^2(2S_0^2 - \tfrac{1}{169}) + \tfrac{1457}{28561}$$

for an arbitrary constant C_0 and $S_0 \leqslant -\frac{1}{26}$ or $S_0 > \frac{1}{26}$. The solution (5.8) after substituting $R(\xi)$ into (5.16) gives the solutions of (5.15) in the form of periodic and solitary waves. In particular, the periodic wave degenerates into a solitary wave,

$$U(\xi) = \tfrac{105}{169}\,\mathrm{ch}^{-4}(\tfrac{1}{2}\xi/\sqrt{13})\,,\quad \xi = x - C_0 t\,, \qquad (5.17)$$

for $d=q=0$, $C_0 = \frac{36}{169}$, $S_0 = -\frac{1}{26}$.

The solution in th form of a solitary wave (5.17) is happy indeed since it has been repeatedly found in a number of papers [14–17].

(5) Let us apply the algorithm to find the periodical solutions expressed by means of Jacobi elliptic functions of the generalized Kawachara equation [15],

$$U_t + 2\alpha_0 U U_x + 3\epsilon_0 U^2 U_x + \beta U_{xxx} = U_{xxxxx}\,. \qquad (5.18)$$

In terms of a travelling wave variable,

$$\beta U_{\xi\xi} - U_{\xi\xi\xi\xi} + \epsilon_0 U^3 + \alpha_0 U^2 - C_0 U + q = 0\,, \qquad (5.19)$$

this equation has the following solution for $\epsilon_0 = 30$ (chosen for convenience),

$$U = C_1 + 2R(\xi)\,,$$
$$C_1 = \tfrac{1}{90}(3\beta + 30S_0 - \alpha_0)\,, \qquad (5.20)$$

where $R(\xi)$ is the solution of the equation for the anharmonic oscillator (4.5) with

$$b = \tfrac{1}{3240}\alpha_0^2 - \tfrac{1}{360}(\beta^2 - 10C_0 + 60S_0^2)\,,$$
$$d = -\tfrac{1}{2160}(\alpha_0 C_0 + 90q - 20C_0 S_0 + 40S_0^3 + 2C_0\beta$$
$$+ 2S_0\beta^2) - \tfrac{1}{291600}\alpha_0^2(\alpha_0 - 30S_0 + 3\beta) + \tfrac{1}{5400}\beta^3\,.$$

The solution of eq. (5.19) is (5.20) in the form of the periodic (cnoidal) wave if $\alpha_0 = \beta = 1$, $\epsilon_0 = 30$, $q = 0$, $\frac{26}{225} \leqslant C_0 < \frac{131}{576} - \frac{7}{64}\sqrt{\frac{7}{15}}$ or $\frac{131}{576} + \frac{7}{64}\sqrt{\frac{7}{15}} < C_0$, S_0 is an arbitrary constant and $R(\xi)$ satisfies eq. (4.5). This solution degenerates into a solitary wave,

$$U(\xi) = \tfrac{1}{15}\,\mathrm{ch}^{-2}(\xi/\sqrt{30})\,,\quad \xi = x - C_0 t\,, \qquad (5.21)$$

for $C_0 = \frac{26}{225}$, $S_0 = -\frac{1}{15}$.

6. Discussion

An algorithm allowing one to find exact solutions of nonlinear nonintegrable equations by means of solving an equation of Painlevé type has been designed on the basis of the singular manifold methods. The Riccati equation and the equation for the anharmonic oscillator were taken as basis references in the examples discussed. Eqs. (1.1) can be expressed by means of other equations from the set of equations (3.1)–(3.3) except the Riccati equation and the equation for the anharmonic oscillator.

The algorithm under study does not ensure the existence of the exact solution of the investigated equation (1.1). This is due to the fact that some expressions depending on the $\phi(x,t)$ are not taken into account when constructing the transformations of the kind (1.5).

The algorithm can be applied to find exact solutions of various nonlinear equations. At our disposal at present there are more than fifty equations whose solutions were obtained by this means. A more restricted algorithm has been used in refs. [16–18] where the exact solutions were sought in terms of the sums of hyperbolic functions. Some of the equations considered can have more generalized types of exact solutions than those in refs. [16–18].

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