Exact solutions of the generalized Kuramoto–Sivashinsky equation

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Transformations for the solutions obtained by the Weiss–Tabor–Carnevale method are used for investigation of several classes of analytical solutions of the generalised Kuramoto–Sivashinsky equation which is nonintegrable by means of the usual inverse scattering transform method.

In 1983 Weiss, Tabor and Carnevale [1] offered the Painlevé property expansion for partial differential equations and demonstrated its efficiency for obtaining Bäcklund transformations and Lax pairs for some types of integrable equations (Korteweg–de Vries, Boussinesq, Kadomtsev–Petviashvili, etc.) using the method of inverse scattering transform.

The essence of the Weiss–Tabor–Carnevale method can be presented in several ways. For instance, we have the nonlinear partial differential equation

$$E(u, u_t, u_x, ..., x, t)=0. \tag{1}$$

If the solution of the equation has the form

$$u=\sum_{j=0}^{\infty} u_j F^{j-p}, \tag{2}$$

where p is an integer, then (1) is supposed to have the Painlevé property. In (2) u_j are coefficients, and $F(x, t)$ is the new function. It is useful to note that the solutions $u(x, t)$ of the form (2) are "single-valued" about the movable, singularity manifolds $F(x, t)=0$.

At first the possibility of a solution of the initial equation of the form (2) was associated with the integrability of eq. (1) by the inverse scattering transform,which corresponds to the idea of Ablowitz, Ramani and Segur [2] about the Painlevé property.

Later it turned out that a certain class of differential equations nonintegrable by the inverse scattering transform can have the form (2). Applying the Weiss–Tabor–Carnevale method one can find some classes of analytical solutions of nonintegrable equations. In this connection the method was applied to get analytical solutions of the Burgers–Korteweg–de Vries equation and the Kawahara equation [3]. Studying the system of equations after substitution of eq. (2) into eq. (1) and setting the expressions at the same orders of $F(x, t)$ equal to zero provided some analytical solutions of the generalized Ginzburg–Landau [4,5] and Kuramoto–Sivashinsky (KS) equations [6–8].

The generalized Kuramoto–Sivashinsky equation occupies a prominent position in describing physical processes in unstable systems [9–12].

This equation has the form

$$u_t+uu_x+\alpha u_{xx}+\beta u_{xxx}+\gamma u_{xxxx}=0, \tag{3}$$

where $u(x, t)$ is a function characterising some physical processes; α, β and γ are constant coefficients; x is the coordinate, t is the time.

Multiplying (3) by $u(x, t)$ and integrating the obtained expression over x from $-\infty$ to ∞ when the condition $\lim_{x\to\pm\infty} u_{mx}=0$ is fulfilled ($m=0, 1, 2, 3$; u_{mx} is the mth order derivative), we find

$$\frac{\partial}{\partial t}\langle u^2\rangle=\alpha\langle u_x^2\rangle-\gamma\langle u_{xx}^2\rangle,$$

$$\langle u_{mx}^2\rangle=\int_{-\infty}^{\infty} u_{mx}^2\,\mathrm{d}x. \tag{4}$$

If $\alpha>0$, $\gamma>0$, eq. (4) shows that the term with the second-order derivative in eq. (3) corresponds to addition of energy to the system, the fourth-order derivative term characterises the dissipation of energy.

Substituting eq. (2) into eq. (3) at $p=3$ and setting the expressions at the same orders of $F(x,t)$ equal to zero one can find the resonances of the solution $u(x,t)$ of the generalised KS equation from the equation [13]

$$(j+1)(j-6)(j^2-13j+60)=0. \tag{5}$$

On account of eq. (5) the KS equation is nonintegrable by the inverse scattering transform method. Taking $u_j\equiv 0$ for $j\geqslant 4$ and substituting eq. (2) into eq. (3) we get the transformation of solutions resembling the Bäcklund transformation for integrable equations by means of the inverse scattering transform [3]:

$$u=60\gamma\frac{\partial^3}{\partial x^3}\ln F+15\beta\frac{\partial^2}{\partial x^2}\ln F$$
$$+\tfrac{15}{76}(16\alpha-\beta^2/\gamma)\frac{\partial\ln F}{\partial x}+u_3. \tag{6}$$

The remaining set of equations containing $u_3(x,t)$ and derivatives F_t, F_x, ..., F_{xxxxx} in the general case happens to be overdetermined, but joint for several classes of functions. In particular the following substitution [14],

$$u_3=c_1=\text{const},$$
$$F(x,t)=c_2+c_3\exp(kx+\omega t), \tag{7}$$

in this overdetermined set of functions suggests solutions of the generalised KS equation in the form of solitary waves and kinks under the following combinations of the parameters of eq. (3):

$$\sigma=\frac{\beta}{\sqrt{\alpha\gamma}}=0,\ \pm\frac{12}{\sqrt{47}},\ \pm\frac{16}{\sqrt{73}},\ \pm 4. \tag{8}$$

Let us demonstrate that the transformation (6) can be directly used to obtain exact solutions of the generalised Kuramoto–Sivashinsky equation.

For this purpose we normalise eqs. (3) and (6) by setting

$$u=\alpha\sqrt{\alpha/\gamma}\,u',\quad x=\sqrt{\gamma/\alpha}\,x',$$
$$t=(\gamma/\alpha^2)t',\quad \sigma=\beta/\sqrt{\alpha\gamma}.$$

Thus eq. (3) has the form

$$u_t+uu_x+u_{xx}+\sigma u_{xxx}+u_{xxxx}=0 \tag{9}$$

and eq. (6) is transformed to the expression

$$u=60\frac{\partial^3\ln F}{\partial x^3}+15\sigma\frac{\partial^2\ln F}{\partial x^2}$$
$$+\tfrac{15}{76}(16-\sigma^2)\frac{\partial\ln F}{\partial x}+u_3. \tag{10}$$

The primes of the variables in eqs. (9) and (10) are omitted. After taking $u(x,t)=v(\xi)$, $\xi=x-c_0t$ (c_0 is the speed of the wave), eq. (9) in the coordinate system of the travelling wave has the form

$$q-c_0v+\tfrac{1}{2}v^2+v_\xi+\sigma v_{\xi\xi}+v_{\xi\xi\xi}=0, \tag{11}$$

where q is the constant of integration.

If $\sigma=4$, $u_3=c_1=\text{const}$, the transformation (10) can be written as

$$v=c_1+R+R'_\xi,\quad R=60\frac{\mathrm{d}^2\ln F}{\mathrm{d}\xi^2}. \tag{12}$$

Substituting eq. (12) into eq. (11) we arrive at

$$Z''_{\xi\xi}+5Z'_\xi+(5-A-\tfrac{1}{5}R)Z+\tfrac{3}{5}\int R'_\xi Z\,\mathrm{d}\xi=0, \tag{13}$$

where

$$Z=R''_{\xi\xi}+\tfrac{1}{10}R^2+AR-\tfrac{5}{2}(1-A^2),$$
$$A=\tfrac{1}{5}(c_1-c_0+1). \tag{14}$$

Multiplying (14) by R'_ξ and integrating the result over ξ, we get

$$2\int R'_\xi Z\,\mathrm{d}\xi=R'^2_\xi+\tfrac{1}{15}R^3+AR^2$$
$$-5(1-A^2)R-\tfrac{10}{3}D. \tag{15}$$

The constant D is connected with the constants c_0, c_1 and q by the relation

$$D=c_1(c_0-\tfrac{1}{2}c_1)+\tfrac{5}{2}(A-5)(1-A^2)-q.$$

It follows from eq. (13) that each solution of the equation

$$R'^2_\xi+\tfrac{1}{15}R^3+AR^2-5(1-A^2)R-\tfrac{10}{3}D=0, \tag{16}$$

after substitution in eq. (12), leads to a solution of the generalised Kuramoto–Sivashinsky equation in the case $\sigma=4$.

Denoting the real values of the relation

$$R^3+15AR^2-75(1-A^2)R-50D=0 \tag{17}$$

as R_1, R_2 and R_3 with $R_1 \geqslant R_2 \geqslant R_3$, the solution of eq. (16) is expressed by means of the elliptic Jacobi function

$$R(\xi)=R_2+(R_1-R_2)\,\mathrm{cn}^2\left(\tfrac{1}{2}\xi\sqrt{\tfrac{1}{15}(R_1-R_3)},\,S\right),$$

$$S^2=\frac{R_1-R_2}{R_1-R_3}\,. \tag{18}$$

The solution (18) is a periodic wave. At $c_0=6$ ($c_1=D=q=0$, $A=-1$) and $c_0=-9$ ($c_1=D=0$, $A=2$, $q=\frac{15}{2}$) the solution (18) is transformed into a solitary wave:

$$R(\xi)=15\,\mathrm{ch}^{-2}(\tfrac{1}{2}\xi). \tag{19}$$

In the general case, the solution of eq. (16) and the solution of the KS equation can be expressed by means of elliptic Weierstrass functions. The solutions of the generalized KS equation can be obtained from the formula

$$v=c_1+R+\frac{1}{\sqrt{15}}\,[50D+75(1-A^2)R$$
$$-15AR^2-R^3]^{1/2}. \tag{20}$$

It is easily seen using eq. (18) that these solutions are also periodic (cnoidal) waves and their amplitudes can be found as the real roots of the cubic equation (17).

To investigate whether the solitary wave obtained after substituting (19) into (20),

$$v=15\,\mathrm{ch}^{-2}(\tfrac{1}{2}\xi)\,[1-\mathrm{th}(\tfrac{1}{2}\xi)], \tag{21}$$

is a soliton, one can arrange the numerical simulation of its interaction with the other solutions of the KS equation by means of a finite difference method.

The calculations indicate that the solution (21) does not change in time and has elastic interaction with other solutions. Therefore the solitary wave (21) satisfying the KS equation from a classical point of view is a soliton [15]. A simulation of the evolution of an initial perturbation, described by eq. (30) with periodic boundary conditions, demonstrated that the initial perturbation irrespective of its form transforms in time in an autostructure. The characteristic peculiarity of this structure at $\sigma=4$ is that it is formed of a "soliton gas" consisting of solitary waves described by eq. (21) [16].

Since eq. (11) is invariant under the transformations

$$v\to -v,\quad c_0\to -c_0,\quad \xi\to -\xi,\quad \sigma\to -\sigma, \tag{i}$$

the solutions of the KS equation at $\sigma=-4$ can be obtained taking into account expressions (12) and (18).

At $\sigma=12/\sqrt{47}$ the transformation (10) for the solutions of the KS equation in the coordinate system of the travelling wave can be written in the form

$$v=60\exp(-2\xi/\sqrt{47})\,\frac{\mathrm{d}}{\mathrm{d}\xi}\Bigg[\exp(\xi/\sqrt{47})$$
$$\times\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\exp(\xi/\sqrt{47})\,\frac{\mathrm{d}\ln F}{\mathrm{d}\xi}\right)\Bigg]+v_3. \tag{22}$$

Let us define the new variable

$$\theta=\exp(-\xi/\sqrt{47}).$$

Then the transformation (22) can be presented in the form

$$v=c_1-\frac{\theta^3}{47\sqrt{47}}\frac{\mathrm{d}R}{\mathrm{d}\theta},\quad R=60\,\frac{\mathrm{d}^2\ln F}{\mathrm{d}\theta^2}\,. \tag{23}$$

Substituting eq. (23) into eq. (11) at $\sigma=12/\sqrt{47}$ and using the new variables,

$$q-c_0v+\tfrac{1}{2}v^2-\frac{\theta}{\sqrt{47}}\frac{\mathrm{d}v}{\mathrm{d}\theta}+\frac{12}{\sqrt{47}}\left(\theta\frac{\mathrm{d}v}{\mathrm{d}\theta}+\theta^2\frac{\mathrm{d}^2v}{\mathrm{d}\theta^2}\right)$$
$$-\frac{1}{47\sqrt{47}}\left(\theta^3\frac{\mathrm{d}^3v}{\mathrm{d}\theta^3}+3\theta^2\frac{\mathrm{d}^2v}{\mathrm{d}\theta^2}+\theta\frac{\mathrm{d}v}{\mathrm{d}\theta}\right)=0, \tag{24}$$

we find, as in the case $\sigma=4$, that each solution $R(\theta)$ of the equation

$$R'^2_\theta+\tfrac{1}{15}(R-B)^3=0\,, \tag{25}$$

(where B is an arbitrary constant) under the condition

$$q=c_1(c_0-\tfrac{1}{2}c_1)$$

leads after its substitution in eq. (23) to a solution of the KS equation in the form of a kink,

$$v=c_1-\frac{120\theta^3}{47\sqrt{47}\,(c_3+\theta)^3}\,, \tag{26}$$

where c_3 is an arbitrary constant.

Because eq. (11) is invariant under the transformation (i) these solutions of the KS equation can be obtained in an evident way at $\sigma=-12/\sqrt{47}$.

At $\sigma=16/\sqrt{73}$ the transformation of the solutions of the KS equation (10) can be presented in the form

$$v=60\exp(-3\xi/\sqrt{73})\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\exp(2\xi/\sqrt{73})\times\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\exp(\xi/\sqrt{73})\frac{\mathrm{d}\ln F}{\mathrm{d}\xi}\right)\right]+v_3. \qquad (27)$$

Setting the new variable

$$\theta=\exp(-\xi/\sqrt{73})$$

we find that the transformation (27) for $v_3=c_1$ can be written as

$$v=c_1-\frac{1}{73\sqrt{73}}\left(\theta^3\frac{\mathrm{d}R}{\mathrm{d}\theta}-\theta^2 R\right),$$

$$R=\frac{\mathrm{d}^2\ln F}{\mathrm{d}\theta^2}. \qquad (28)$$

Substituting this expression into eq. (11) using the new variables we obtain

$$q-c_0v+\tfrac{1}{2}v^2-\frac{\theta}{\sqrt{73}}\frac{\mathrm{d}v}{\mathrm{d}\theta}+\frac{16}{73\sqrt{73}}\left(\theta\frac{\mathrm{d}v}{\mathrm{d}\theta}+\theta^2\frac{\mathrm{d}^2v}{\mathrm{d}\theta^2}\right)-\frac{1}{73\sqrt{73}}\left(\theta\frac{\mathrm{d}v}{\mathrm{d}\theta}+3\theta^2\frac{\mathrm{d}^2v}{\mathrm{d}\theta^2}+\theta^3\frac{\mathrm{d}^3v}{\mathrm{d}\theta^3}\right)=0, \qquad (29)$$

we find that all the solutions of the equation

$$R_\theta'^2+\tfrac{1}{15}R^3=0 \qquad (30)$$

under the condition

$$q=c_1(c_0-\tfrac{1}{2}c_1), \quad c_0=c_1-\frac{90}{73\sqrt{73}}$$

satisfy eq. (29) and consequently formula (28) gives the solution of the KS equation for $\sigma=16/\sqrt{73}$. They have the form of kinks,

$$v=c_1-\frac{60}{73\sqrt{73}}\left(\frac{\theta^2}{(c_3+\theta)^2}+\frac{2\theta^3}{(c_3+\theta)^3}\right). \qquad (31)$$

The exact solution of the KS equation at $\sigma=0$ can be found analogously. For this case the transformation of the solution (10) can be presented in the form

$$v=c_1+\frac{\mathrm{i}}{19\sqrt{19}}\frac{\mathrm{d}}{\mathrm{d}\theta}\theta^3R,$$

$$\theta=\exp(-\mathrm{i}\xi/\sqrt{19}), \quad R=50\frac{\mathrm{d}^2\ln F}{\mathrm{d}\theta^2}. \qquad (32)$$

After substitution of eq. (32) into eq. (11), using the new variables, and taking account of

$$q=c_1(c_0-\tfrac{1}{2}c_1), \quad c_0=c_1-\frac{30\mathrm{i}}{19\sqrt{19}},$$

we find that each solution of the equation

$$R_\theta'^2+\tfrac{1}{15}R^3=0$$

gives a solution of the KS equation at $\sigma=0$ by formula (32),

$$u=c_1-\frac{60\mathrm{i}}{19\sqrt{19}}\left(\frac{3\theta^2}{(c_3+\theta)^2}-\frac{2\theta^3}{(c_3+\theta)^3}\right).$$

Setting in (33)

$$\theta=\exp(-\mathrm{i}\xi/\sqrt{19}), \quad c_3=1, \qquad (33)$$

we arrive at the solution of eq. (3) given in refs. [6,8].

Exact solutions of the generalised Kuramoto–Sivashinsky equation can also be obtained if we rewrite the transformation (6) taking into account that $u_4(x,t)$ is a function of the derivatives of $F(x,t)$ with respect to x and t [14],

$$u=30\frac{\partial^3}{\partial x^3}\ln\frac{F^2}{F_x}+\tfrac{15}{2}\sigma\frac{\partial^2}{\partial x^2}\ln\frac{F^2}{F_x}+\tfrac{15}{152}(16-\sigma^2)\frac{\partial}{\partial x}\ln\frac{F^2}{F_x}-\frac{F_t}{F_x}+\tfrac{5}{2}\sigma\{F;x\}+15\frac{\partial}{\partial x}\{F;x\}+\tfrac{1}{76}\sigma(7-\tfrac{13}{8}\sigma^2). \qquad (34)$$

Substituting eq. (34) into eq. (6) and setting the expressions at the same powers of $F(x,t)$ equal to zero, we obtain that $F(x,t)$ satisfies the obtained set of equations if

$$\mathrm{Sh}=\{F;x\}=m, \quad F_t/F_x=-c_0, \qquad (35)$$

where c_0 is an arbitrary constant, Sh is the Schwarz derivative. The constants m can be defined with respect to σ,

$$\sigma=4: \quad m_1=\tfrac{1}{2}, \quad m_2=-\tfrac{1}{2},$$

$$\sigma=0: \quad m_1=\tfrac{1}{38}, \quad m_2=-\tfrac{11}{38},$$

$$\sigma=12/\sqrt{47}: \quad m_1=-\tfrac{1}{94},$$

$$\sigma=16/\sqrt{73}: \quad m_1=-\tfrac{1}{146}. \qquad (36)$$

The general solution of eq. (35) can be written in the form [8]

$$F(x,t)=\frac{ay_1+by_2}{dy_1+ey_2}, \quad ae\neq bd, \qquad (37)$$

where a, b, d and e are arbitrary constants and y_1 and y_2 are two independent solutions of the linear equation

$$y''+\tfrac{1}{2}my=0.$$

The exact solutions of the generalised Kuramoto–Sivashinsky equation can be obtained after substituting eq. (37) into eq. (34).

In this paper analytical solutions of the KS equation have been obtained with the help of the transformations (10) allowing one to obtain the Painlevé-type equations (16), (25), (30). It is important to note that all these equations have resonances at $j=-1$ and $j=6$ [17], corresponding to resonances (5) of the initial KS equation (3).

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