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EXACT SOLITON SOLUTIONS OF THE GENERALIZED EVOLUTION EQUATION OF WAVE DYNAMICS*

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A Backlund transformation is proposed for the generalized evolution equation of gas dynamics, by means of which exact soliton solutions of this equation are obtained.

In recent years, a non-linear fourth-order equation has been used to describe a number of wave processes. In the general case, this takes the form

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}=\alpha\frac{\partial^2 u}{\partial x^2}+\gamma\frac{\partial^4 u}{\partial x^4} \tag{0.1}$$

Here α, β and γ are constant coefficients, $u(x, t)$ is a function that characterizes the physical process: mixing, the thickness of a film, concentration, etc.

With $\alpha\neq 0, \beta=\gamma=0$ Eq.(0.1) is the Burgers equation, which, in the simplest case, models the formation of shock waves in gas dynamics /1/. Using a Cole-Hopf transformation /2, 3/

$$u(x, t)=-2\alpha\partial\ln F/\partial x \tag{0.2}$$

the Burgers equation is transformed into a linear heat conduction equation with respect to the function $F(x, t)$. When $\alpha=\gamma=0, \beta\neq 0$ Eq.(0.1) is well-known as the Korteveg-de Vries (KdV) equation, which describes solitons (localized non-linear waves) /4/.

Using the Miura transformation /5, 6/

$$u(x, t)=12\beta\partial^2\ln F/\partial x^2 \tag{0.3}$$

the KdV equation reduces to an equation for $F(x, t)$ which has a quadratic form, from which Hirota /7/ found exact single- and multi-soliton solutions of the KdV equation.

Below, we will consider Eq.(0.1) with values of the coefficients α, β and γ different from zero.

1. The Backlund transformation for Eq.(0.1). We write the solution of (0.1) in the form of the following sum:

$$u(x, t)=\sum_{j=0}^{\infty}u_j(x, t)F^{j-3}(x, t) \tag{1.1}$$

Substituting (1.1) into (0.1) and equating terms with the same powers of $F(x, t)$ we get a series of equalities:

$$u_0=-120\gamma F_x^3,\ u_1=-15\beta F_x^2+180\gamma F_xF_{xx}$$
$$u_2=(15/76)(\beta^2/\gamma-16\alpha)F_x+15\beta F_{xx}-60\gamma F_{xxx} \tag{1.2}$$

We can write the equation that contains the coefficient $u_3(x, t)$ and partial derivatives of $F(x, t)$ (denoted by F_t, F_x, F_{xx} etc.) in the form

$$F_t+u_3F_x+\frac{\beta}{76\gamma}\left(\frac{13\beta^2}{8\gamma}-7\alpha\right)F_x+\frac{15}{152}\left(\frac{\beta^2}{\gamma}-16\alpha\right)F_{xx}+$$
$$5\beta F_{xxx}-15\gamma F_{xxxx}-\frac{15}{4}\beta F_{xx}^2F_x^{-1}+30\gamma F_{xx}F_{xxx}F_x^{-1}-$$
$$15\gamma F_{xx}^3F_x^{-2}=0 \tag{1.3}$$

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We can set up a recurrence formula for finding the coefficients of $u_j(x, t)$ with $j \geqslant 4$*. (*N.A. Kudryashov, The Bucklund transformation for the viscoelastic wave equation in dispersive media: Preprint 007-87. MIFI, Moscow, 1987.) It turned out, however, that u_4 is not determined from this formula, and hence we must put $u_j = 0$ for $j \geqslant 4$. Setting $u_4 = 0$ in the recurrence formula and taking account of expression (1.2), we get an equation for $F(x, t)$:

$$F_{xt} - F_t F_{xx} F_x^{-1} + \frac{1}{722}\left(11\frac{\alpha^2}{\gamma} - \frac{87}{8}\frac{\alpha\beta^2}{\gamma^2} + \frac{131}{64}\frac{\beta^4}{\gamma^3}\right)F_x + \frac{5}{152}\left(\frac{\beta^2}{\gamma} - 16\alpha\right)F_{xxx} + \frac{5}{4}\beta F_{xxxx} - 3\gamma F_{xxxxx} + 5\beta F_{xx}F_{xxx}F_x^{-1} + \frac{15}{4}\beta F_{xx}^3 F_x^{-2} - \frac{15}{304}\left(\frac{\beta^2}{\gamma} - 16\alpha\right)F_{xx}^2 F_x^{-1} + 15\gamma F_{xx}F_{xxxx}F_x^{-1} - 45\gamma F_{xx}^2 F_{xxx} F_x^{-2} + \frac{45}{2}\gamma F_{xx}^4 F_x^{-3} + 10\gamma F_{xxx}^2 F_x^{-1} = 0 \tag{1.4}$$

In addition, the recurrence formula gives the following equation that contains u_1, u_2, and u_3:

$$\frac{\partial u_1}{\partial t} - u_2 F_t - u_2 u_3 F_x + u_2\frac{\partial u_2}{\partial x} + \frac{\partial}{\partial x}(u_1 u_3) = Lu_1 - L(u_2 F) + F L u_2$$
$$\frac{\partial u_2}{\partial t} + \frac{\partial (u_2 u_3)}{\partial x} = Lu_2, \quad \frac{\partial u_3}{\partial t} + u_3\frac{\partial u_3}{\partial x} = Lu_3$$
$$L = \alpha\frac{\partial^2}{\partial x^2} - \beta\frac{\partial^3}{\partial x^3} + \gamma\frac{\partial^4}{\partial x^4} \tag{1.5}$$

Since $u_j(x, t) = 0$ with $j \geqslant 4$, taking (1.2) into account we can write the solution of (0.1) by means of the formula

$$u(x, t) = \frac{15}{76}\left(\frac{\beta^2}{\gamma} - 16\alpha\right)\frac{\partial}{\partial x}\ln F + 15\beta\frac{\partial^2}{\partial x^2}\ln F - 60\gamma\frac{\partial^3}{\partial x^3}\ln F + u_3(x, t) \tag{1.6}$$

The last equation of (1.5) for the coefficient $u_3(x, t)$ is identical in form with the initial Eq.(0.1), and so (1.6) can be used to transform the solution of (0.1).

For the Burgers-Korteveg-de Vries(BKdV) equation, which is obtained from (0.1), if $\alpha \neq 0$, $\beta \neq 0$, $\gamma = 0$ the transformation of the equations (analogous to (1.6)) takes the form

$$u(x, t) = -\frac{12}{5}\alpha\frac{\partial}{\partial x}\ln F + 12\beta\frac{\partial^2}{\partial x^2}\ln F + u_3(x, t) \tag{1.7}$$

As also when $\gamma \neq 0$, the coefficients $u_j = 0$ when $j \geqslant 4$. The coefficient $u_3(x, t)$ is connected with $F(x, t)$ by the following equation:

$$F_t + u_3 F_x - \frac{\alpha^2}{25\beta}F_x - \frac{6\alpha}{5}F_{xx} + 4\beta F_{xxx} - 3\beta F_{xx}^2 F_x^{-1} = 0 \tag{1.8}$$

For $F(x, t)$ with $\gamma = 0$ we obtain the equation

$$F_{xt} - F_t F_{xx} F_x^{-1} - \frac{\alpha^3}{125\beta^2}F_x - \frac{2\alpha}{5}F_{xxx} + \beta F_{xxxx} + \frac{3\alpha}{5}F_{xx}^2 F_x^{-1} - 4\beta F_{xx}F_{xxx}F_x^{-1} + 3\beta F_{xx}^3 F_x^{-2} = 0 \tag{1.9}$$

In this case, the coefficient $u_3(x, t)$ also obeys (0.1) if we set $\gamma = 0$. When $\alpha = 0$ and $u_3(x, t) = 0$, transformation (1.7) becomes the Miura transformation (1.3) for the KdV equation.

2. Exact solutions of the BKdV equation.

Using (1.7), we will find a solution of the BKdV equation (Eq.(0.1) with $\gamma = 0$).

With $u_3(x, t) = 0$, we get an equation for $F(x, t)$, which can be represented by setting the quadratic form equal to zero

$$3\beta G_3 - G_1 G_2 + F\frac{\partial G_1}{\partial x} + \frac{\alpha}{5}F_x\frac{\partial G_2}{\partial x} = 0$$
$$G_1 = F_t - \alpha F_{xx} + \beta F_{xxx}$$
$$G_2 = F_x + \frac{\alpha}{5\beta}F_{xxx}, \quad G_3 = F_{xx}^2 - F_x F_{xx} \tag{2.1}$$

The left-hand side of (2.1) with $\alpha = 0$ is identical with the quadratic form that was applied to determine the soliton solutions of the KdV equation /5, 6/.

The function $F(x, t)$ that obeys the conditions $G_1 = c_0, G_2 = 0, G_3 = 0$ (c_0 is a constant) is a solution of Eq.(2.1). Setting $c_0 = 0$, we find

$$F(x, t) = c_1 + c_2 e^{kx-\omega t} \tag{2.2}$$

$$k = -\frac{\alpha}{5\beta}, \quad \omega = \frac{6\alpha^3}{125\beta^2} \tag{2.3}$$

where c_1 and c_2 are constants.

Substituting (2.2) into (1.7) with $u_3(x, t) = 0$ and setting $c_1 = c_2 = 1$, we can write the solution of the BKdV equation in the form

$$u(x, t) = {}^3/_5 \alpha k\,[U^2(x, t) + 2U(x, t) - 3] \qquad U(x, t) = \mathrm{th}\,[{}^1/_2\,(kx - \omega t)] \tag{2.4}$$

The equation we have obtained has the form of a shock-wave, which is characteristic for solutions of the Burgers equation. As was proved in /8/, the BKdV equation has shock-wave solutions with monotonic profiles (for $\alpha > \alpha_*$) and oscillatory profiles (for $(\alpha < \alpha_*)$. For (2.4) we have $\alpha_* = \sqrt{0.96}\alpha$, which corresponds to a monotonic profile of the shock-wave front.

Note that if we immediately look for a solution of (2.1) in the form of (2.2) with unknowns k, ω, then as a result of the substitution into (2.1) we obtain the values (2.3) for k and ω. Analogous values of k and ω are obtained if we substitute (2.2) into (1.8) and (1.9), having set $u_3(x, t) = 0$ in (1.8).

3. Solution of Eq.(0.1) for $\beta = 0$, $\alpha \neq 0$, $\gamma \neq 0$. For $\beta = 0$ and $u_3(x, t) = 0$, substituting (1.6) into (0.1) we obtain the result that the following cubic form is equal to zero:

$$\gamma F F_{xx} G_4 - \frac{\alpha}{19} F^2 G_4 - 2\gamma F_x^2 G_4 + \gamma F F_x \frac{\partial G_4}{\partial x} - \frac{30}{19}\alpha\gamma F G_3 - 5\gamma^2 F G_6 - 30\gamma^2 G_7 + \gamma L^1 G_4 + \frac{\alpha}{19} L^1 G_5 + \gamma L^2 G_5 = 0 \tag{3.1}$$

$$G_4 = F_t - \frac{30}{19}\alpha F_{xx}, \quad G_5 = \frac{11}{19}\alpha F - \gamma F_{xx}$$

$$G_6 = F_x F_{xxxxx} + 3F_{xx}F_{xxxx} - 4F_{xxx}^2$$

$$G_7 = 2F_x F_{xx} F_{xxx} - F_x^2 F_{xxxx} - F_{xx}^3$$

$$L^k = F F_x \frac{\partial^k}{\partial x^k} - F^2 \frac{\partial^{k+2}}{\partial x^{k+2}}$$

The constant of integration is taken to be equal to zero in (3.1). It can be seen from the cubic form (3.1) that the function $F(x, t)$ is a solution of (3.1) if it obeys the conditions $G_4 = 0$, $G_5 = c_3$, $G_3 = G_6 = G_7 = 0$ (c_3 is a constant). These conditions are satisfied for (2.2) with

$$k = \pm\sqrt{11\alpha/(19\gamma)}, \quad \omega = (30/19)\alpha k^2 \tag{3.2}$$

Setting c_1 and $c_2 = 1$ and substituting (2.2) and (3.2) into (1.6), with $\beta = 0$ we find a solution of Eq.(0.1)

$$u(x, t) = \frac{15}{19}\alpha k\,[9U(x, t) - 11U^3(x, t) - 2], \qquad U(x, t) = \mathrm{th}\left[\frac{k}{2}\left(x - \frac{30}{19}\alpha k t\right)\right] \tag{3.3}$$

For $\alpha = -1$ and $\gamma = -1$, solution (3.3) is identical with the Kuramoto solution that has been proposed for describing concentration waves in chemical reactions /9/. Expressions (3.3) is also known as an exact soliton solution of the equation that describes the sliding of a film down an inclined plane /10/. Expression (2.2), (3.2), for $F(x, t)$ obeys the system of Eqs.(1.3) and (1.4) in which $u_3(x, t) = 0$ and $\beta = 0$. This system may also be used to find the values of k and ω in expression (2.2) for $F(x, t)$.

For ω we obtain the second relationship in (3.2), and for k we find

$$k_{1,2} = \pm\sqrt{11\alpha/(19\gamma)}, \quad k_{3,4} = \pm i\sqrt{\alpha/(19\gamma)} \tag{3.4}$$

The values of k_1 and k_2 lead to the solution cited above.

4. Solution of Eq.(0.1) for $\alpha \neq 0$, $\beta \neq 0$, $\gamma \neq 0$. The substitution of (1.6) with $u_3(x, t) = 0$ into (0.1) leads to a cumbersome expression. So we will look for $F(x, t)$ in the form of (2.2). Substituting (2.2) into (1.3) and (1.4) with $u_3(x, t) = 0$, we get an algebraic equation for ω and k:

$$\omega=\frac{5\beta k}{4}\left[k^2+\frac{3}{38\beta}\left(\frac{\beta^2}{\gamma}-16\alpha\right)k+\frac{1}{95\gamma}\left(\frac{13\beta^2}{8\gamma}-7\alpha\right)\right. \tag{4.1}$$

$$k^4+\frac{5}{152\gamma}\left(\frac{\beta^2}{\gamma}-16\alpha\right)k^2-\frac{1}{361}\left(11\frac{\alpha^2}{\gamma^2}-\frac{87}{8}\frac{\alpha\beta^2}{\gamma^3}+\frac{131}{64}\frac{\beta^4}{\gamma^4}\right)=0 \tag{4.2}$$

We find $k_1,\ldots,k_4$ from the quartic Eq.(4.2). From the first expression in (1.5), taking account of (2.2) and (4.2), we find

$$\beta\left(\frac{\beta^2}{\gamma}-16\alpha\right)\left[5k^2+\frac{1}{19\gamma}\left(\frac{13\beta^2}{8\gamma}-7\alpha\right)\right]=0 \tag{4.3}$$

The algebraic equation for k that is obtained from the second equation in (1.5) is a consequence of (4.2) and (4.3).

Thus, the function $F(x,t)$, defined by (2.2) with ω and k calculated from (4.1)-(4.3) is a solution of the system of Eqs.(1.3)-(1.6). Since expressions (1.3)-(1.6) are obtained as a result of substituting (1.1) into (0.1) and equating terms with the same powers of $F(x,t)$, it follows from the fact that (1.3)-(1.6) vanish that the function (1.6), where $F(x,t)$ is determined from (2.2) with ω and k calculated from (4.1)-(4.3), is a solution of the initial Eq.(0.1).

It can be seen from (4.1)-(4.3) that the values of ω and k effectively depend on the parameters α, β and γ.

The solution $u(x,t)$ of the initial Eq.(0.1) is found if we substitute (2.2) into (1.6). Setting $c_1=c_2=1$ in (2.2), as we did before, and substituting $F(x,t)$ into (1.6), we find

$$u(x,t)=15k_0^2\left[\frac{\beta}{4}+\gamma k_0U(x,t)\right]\mathrm{ch}^{-2}\left[\frac{1}{2}(k_0x-\omega_0t)\right]+\frac{15k_0}{152}\left(\frac{\beta^2}{\gamma}-16\alpha\right)[1+U(x,t)],\quad U(x,t)=\mathrm{th}\left[\frac{1}{2}(k_0x-\omega_0t)\right] \tag{4.4}$$

Here k_0 is the value of one of the real roots of the system of Eqs.(4.2), (4.3) and $\omega_0=\omega(k_0)$ is calculated from (4.1).

Consider an actual analytic solution of (0.1). For $\beta=0$ we obtain the result that Eq.(1.5) is a consequence of (1.3) and (1.4). From the algebraic Eq.(4.2) for k we obtain the values (3.4). As a result solution (4.4) becomes the Kuramoto solution (3.3). When $\beta^2=16\alpha\gamma$, the left-hand side of (4.3) becomes zero, since here (1.5) is also a consequence of (1.3) and (1.4). From (4.1) and (4.2) we obtain

$$k^2_{1,2,3,4}=\pm\alpha/\gamma,\quad \omega=5k\sqrt{\alpha\gamma}\,|(k^2+\alpha/5\gamma)$$

For $k_0=\sqrt{\alpha/\gamma}$, $\omega_0=\omega(k_0)=6\alpha^2\gamma^{-1}$, the solution $u(x,t)$ of Eq.(0.1) is written in the form

$$u(x,t)=15\alpha\sqrt{\frac{\alpha}{\gamma}}\,\mathrm{ch}^{-2}\left\{\frac{1}{2}\sqrt{\frac{\alpha}{\gamma}}\left(x-\frac{6\alpha\sqrt{\alpha}}{\sqrt{\gamma}}t\right)\right\}\times\left[1+\mathrm{th}\left\{\frac{1}{2}\sqrt{\frac{\alpha}{\gamma}}\left(x-\frac{6\alpha\sqrt{\alpha}}{\sqrt{\gamma}}t\right)\right\}\right] \tag{4.5}$$

Solution (4.5) has a single maximum, which is equal to $(160/9)\sqrt{\alpha^3/\gamma}$. If $(x-6\alpha\sqrt{\alpha/\gamma}t)\to\pm\infty$, then $u(x,t)\to 0$. There are also other analytic solutions of (0.1). Equating the values of k^2 from (4.2) and (4.3) we obtain a quartic equation for $\beta/\sqrt{\alpha\gamma}$, from which

$$\beta_{1,2}=\pm19\sqrt{\alpha\gamma/47},\quad \beta_{3,4}=\pm16\sqrt{\alpha\gamma/73} \tag{4.6}$$

We find the corresponding values of k from (4.2):

$$k_{1,2}=\pm\sqrt{\alpha/(47\gamma)},\quad k_{3,4}=\pm\sqrt{\alpha/(73\gamma)} \tag{4.7}$$

Substituting $k_{1,2}$ and $k_{3,4}$ into (4.1), we obtain

$$\omega_1=-60\alpha^2/(47^2\gamma),\quad \omega_2=-90\alpha^2/(73^2\gamma) \tag{4.8}$$

Expression (4.4) is an analytic solution of (0.1) with values of k and ω corresponding to (4.7) and (4.8) provided that the coefficient β is connected with α and γ by (4.6).

We will consider the solutions (4.4) and (4.5) that we have obtained for (0.1). Fig.1 shows the solutions $u(x,t)$ in the travelling-wave coordinate system $\xi=x=\omega_0k_0^{-1}t$ with $\alpha=\gamma=1$ and $\beta=-4,0,4$; real values of k_0 are found from formula (4.2) taking account of the equality (4.3). For $\beta=-4$ ($\beta^2=16\alpha\gamma$) the solution $u(x,t)$ has a single maximum corresponding to a solitary wave, which is completely smoothed as $\xi\to\pm\infty$ (in fact, even with $|\xi|=5$). In the case $\beta=0$, the solution has a maximum and a minimum. With $\beta=4$ the solution $u(\xi)$

describes a solitary wave with one maximum and one level, which is equal to zero for fairly large $|\xi|$. As the absolute value of β increases, the wave becomes narrower. For the case $\beta^2 = 16\alpha\gamma$ with a fixed value of γ, as α (and, consequently, β) increases, the increase in wave amplitude and decrease in its width are also characteristic. The velocity of the solitary wave is proportional to $\alpha^{3/2}$, its value at $\xi = 0$, and the amplitude also $\sim\alpha^{3/2}$. Consequently, solitary waves with a large amplitude, described by (0.1), will travel with a high velocity.

Analysis of the stability of the motion described by (0.1) with respect to small perturbations $u' \sim e^{ikx+\omega t}$ leads to the following dispersion relationship:

$$\omega = -i\beta k^3 - k^2(\alpha - \gamma k^2) \qquad (4.9)$$

It can be seen from (4.7) that the amplitude of the wave with $k = \sqrt{\alpha/\gamma}$ neither increases nor decreases with time. This corresponds to a solitary wave described by solution (4.5). The long-wave oscillations ($k < \sqrt{\alpha/\gamma}$) with $\alpha < 0$ and $\gamma < 0$ in the system decay in accordance with (4.9), and the short-wave oscillations ($k > \sqrt{\alpha/\gamma}$), on the other hand, increase. This, however, does not contradict the existence of waves (4.4) with $k \neq \sqrt{\alpha/\gamma}$, since such waves connect different levels of the solution $u(x, t)$ as $x \to \pm\infty$. If we take into account the fact that the term αu_{xx} ($\alpha > 0$, $\gamma > 0$) is responsible for dissipating the energy of the wave, and the term γu_{xxxx} is responsible for pumping it, the potential energies of the different levels in the solution also leads to precisely the existence of waves with $k \neq \sqrt{\alpha/\gamma}$.

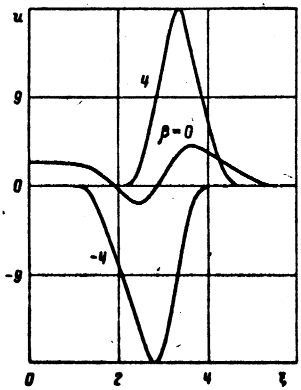

Fig.1

The modelling of the propagation of a solitary wave, carried out on the basis of a numerical solution of (0.1) by changing to a difference scheme, has enabled us to establish that the solitary wave described by (4.5) interacts elastically with other interactions and, consequently, is a soliton.

Comparison of the analytic solutions to (0.1) obtained in this paper with numerical solutions has demonstrated the good agreement of the results.

The procedure for finding exact solutions of (0.1) can be generalized to the case of partial differential equations with non-linearity of BKdV or higher order. There are a number of facts that point to the existence of solutions, analogous to those proposed for (0.1) in this paper, for higher-order equations.

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