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Fourth-order analogies to the Painlevé equations

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Abstract
Using the compatibility condition for the Painlevé equations, several new fourth-order ordinary differential equations (ODEs) that are analogies of the Painlevé equations are found. The isomonodromic linear problems for these equations are given. Special solutions of the fourth-order ODEs found are discussed. The Painlevé test is applied to investigate several fourth-order ODEs.

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1. Introduction

In recent years there has been a renewal of interest in the theory of the Painlevé equations [1]. This interest stemmed from the observation by Ablowitz and Segur [2–4] that reductions of a nonlinear partial differential equation (PDE) of the soliton type give rise to ordinary differential equations (ODEs) whose movable singularities are only poles. Using this idea, a number of the Painlevé equations were obtained as reductions of soliton equations.

The Painlevé equations were first found by Painlevé and his collaborators more than a century ago when Painlevé and his school began an investigation of the nonlinear second-order ODEs class. They wanted to solve two different problems: to classify second-order differential equations of a certain form on the basis of their possible singularities of solutions, and to identify second-order differential equations which define new functions. The latter problem was formulated by Fuchs and Poincaré in 1884. However, Fuchs and Poincaré did not find any new functions because they considered the first-order ODE class.

Painlevé and his school found 50 canonical classes of equations whose solutions have no movable critical points. Furthermore, they also showed that among 50 equations there are exactly six second-order ODEs that define new functions. At the present time these new functions are called Painlevé transcendents; and equations with general solutions in the form of these transcendents are called Painlevé equations. These six Painlevé equations were first discovered from strictly mathematical investigations but these equations have recently appeared in several physical applications [5].

The results of Painlevé and his school led to the following problems: to classify other types of nonlinear differential equations and to find higher-order differential equations that define new transcendental functions with respect to constants of integration. Recently an attempt has been made to find new functions, other than the Painlevé transcendents, again determined by nonlinear ODEs. With this aim several hierarchies of ODEs were introduced using hierarchies of nonlinear PDEs that are solvable by the inverse scattering transform [6–10].

The aim of this paper is to present several new fourth-order ODEs that are analogies to the Painlevé equations. These equations will be found using the compatibility condition for the Painlevé equations. The isomonodromic linear problems for these equations will also be given. The equations found have special solutions in the form of transcendents and we hope the general solutions of these equations are also transcendents.

The outline of this paper is as follows. The method applied to find higher-order ODEs that are analogies of Painlevé equations is discussed in section 2. Fourth-order ODEs with linear potential are given in section 3. Equations corresponding to the quadratic potential are presented in section 4. Special solutions of these equations are also discussed. The application of the Painlevé test for studying equations with potentials (2.3) and (2.4) is presented in section 5.

2. Method applied

Garnier [11] found that five out of six Painlevé equations can be presented as isomonodromic linear problems. These problems can be used for solving the Painlevé equations by the inverse monodromy transform [12, 13]. The compatibility condition of the isomonodromic linear problem can be written in general form as the following equation [14, 15]:

$$\omega U_\lambda = 4UA_x + 2U_xA - A_{xxx} \tag{2.1}$$

where $U \equiv U(x, \lambda)$ is a potential, $\omega \equiv \omega(\lambda)$ is a dependence on λ and $A \equiv A(x, \lambda)$ is a function of x and λ.

It is known that five of six Painlevé equations can be obtained from equation (2.1) if we look for $A(x, \lambda)$ in the form

$$A(x, \lambda) = a_1(x) + a_0(x)\lambda. \tag{2.2}$$

Assuming $U(x, \lambda)$ in the form

$$U(x, \lambda) = P(x) - \lambda \tag{2.3}$$

we have the first Painlevé equation from equation (2.1) at $\omega(\lambda) = 1$. In the case of $\omega(\lambda) = \lambda$ we get P_{34} and we have the special case of the third Painlevé equation at $\omega(\lambda) = \lambda^2$.

Assuming the potential $U(x, \lambda)$ in the form

$$U(x, \lambda) = P(x) - 2\lambda y(x) + \lambda^2 \tag{2.4}$$

one can find the second Painlevé equation at $\omega(x) = 1$, the fourth Painlevé equation at $\omega(\lambda) = \lambda$, the third Painlevé equation in the case $\omega(\lambda) = \lambda^2$ and the fifth Painlevé equation at $\omega(\lambda) = \lambda(\lambda - \lambda_0)$.

Let us remark that equation (2.1) is equivalent to two isomonodromic linear problems. One can see that the compatibility condition in the form

$$(\Psi_{xx})_\lambda = (\Psi_\lambda)_{xx} \tag{2.5}$$

for the system of equations

$$\Psi_{xx} = (P(x) - \lambda)\Psi \qquad \omega(\lambda)\Psi_\lambda = 2A(x, \lambda)\Psi_x - A_x(x, \lambda)\Psi \tag{2.6}$$

leads to equation (2.1) if we take the linear potential (2.3) in the last equation.

On the other hand, the compatibility condition (2.5) of the isomonodromic linear problem

$$\Psi_{xx} = 2(\lambda - y)\Psi_x + Q(x)\Psi \qquad \omega(\lambda)\Psi_\lambda = C(x,\lambda)\Psi_x + D(x,\lambda)\Psi \tag{2.7}$$

can also be presented as equation (2.1) at the quadratic potential (2.4) if we assume

$$Q(x) = P(x) - y_x - y^2, \qquad C(x,\lambda) = 2A(x,\lambda) \tag{2.8}$$

and

$$D(x,\lambda) = 2A(x,\lambda)(y-\lambda) + \omega(\lambda)x - A_x(x,\lambda). \tag{2.9}$$

One can look for higher-order ODEs that are analogies to the Painlevé equations taking into account the isomonodromic linear problems (2.6) and (2.7) [16]. However, we can look for the higher-order analogies to the Painlevé equations if we use equation (2.1), potentials (2.3), (2.4) and $A(x,\lambda)$ in the form

$$A(x,\lambda) = \sum_{i=0}^{n} a_i(x)\lambda^{n-i}. \tag{2.10}$$

In this paper we are going to apply the formula (2.10) at $n = 2$ which allows one to find several new fourth-order ODEs with the general solutions in the form of the transcendental functions with respect to constants of integration.

3. Fourth-order equations with potential (2.3)

Let us find fourth-order ODEs with potential (2.3). Assuming $n = 2$ in equation (2.10), we have

$$A(x,\lambda) = a_2(x) + a_1(x)\lambda + a_0(x)\lambda^2. \tag{3.1}$$

3.1. Case

$\omega(\lambda) = 2\omega = \text{const}$. Substituting equations (2.3) and (3.1) into (2.1) and equating expressions at different λ to zero we have

$$a_0(x) = c_0 \tag{3.2}$$

$$a_1(x) = \tfrac{1}{2}c_0P(x) - \tfrac{1}{4}c_1 \tag{3.3}$$

$$a_2(x) = -\tfrac{1}{8}c_1P(x) - \tfrac{1}{8}c_0P_{xx} + \tfrac{3}{8}c_0P^2 - \tfrac{1}{4}c_2 \tag{3.4}$$

$$a_{2,xxx} - 2P_xa_2 - 4Pa_{2,x} - 2\omega = 0. \tag{3.5}$$

Here c_0, c_1, c_2 and later c_3 and c_4 are constants of integration. Substituting equation (3.4) into (3.5) after integration we obtain

$$P_{xxxx} - 10PP_{xx} - 5P_x^2 + 10P^3 + \frac{c_1}{c_0}P_{xx} - \frac{3c_1}{c_0}P^2 + 16\frac{\omega x}{c_0} - \frac{4c_2}{c_0}P + c_3 = 0. \tag{3.6}$$

Let us denote $P(x) \equiv y(x)$ and $\omega = -\alpha c_0/16$, $c_1 = \beta c_0$, $c_2 = \mu c_0/4$, $c_3 = \delta$. Then equation (3.6) can be presented in the form

$$y_{xxxx} - 10yy_{xx} - 5y_x^2 + 10y^3 + \beta y_{xx} - 3\beta y^2 - \alpha x = 0. \tag{3.7}$$

We have taken $\mu = 0$ and $\delta = 0$ in equation (3.7) taking into account the change of variables x and y. Equation (3.7) is the generalization of the second member of the first Painlevé hierarchy that was found in [7]. Equation (3.7) at $\beta = 0$ was studied intensively in [17–21]. This equation passes the Painlevé test [17]. The general solution of this equation is an essentially transcendental function with respect to constants of integration [20,21]. There are the Bäcklund transformations for solutions of this equation [19]. At $\beta \neq 0$ equation (3.10) was found in a recent work [22].

The isomonodromic linear problem corresponding to equation (3.7) can be presented in the form [10, 17]

$$\Psi_{xx} = (y - \lambda)\Psi \qquad 8\alpha\Psi_\lambda = 2A_1(x, \lambda)\Psi_x - A_{1,x}\Psi \tag{3.8}$$

where $A_1(x, \lambda)$ is determined by the formula

$$A_1(x, \lambda) = y_{xx} - 3y^2 + \beta y + 2(\beta - 2y)\lambda - 8\lambda^2. \tag{3.9}$$

The system of equations (3.8) can be used to solve the Cauchy problem for equation (3.7) by the inverse monodromy transform.

3.2. Case

$\omega(\lambda) = 2\omega\lambda$. Substituting equations (2.3) and (3.1) into (2.1), we get $a_0(x)$ and $a_1(x)$ in the form (3.2) and (3.3). As a result of substituting these expressions into other equations we obtain the following system of equations:

$$a_2(x) + \tfrac{1}{8}c_1 P + \tfrac{1}{8}c_0 P_{xx} - \tfrac{1}{2}\omega x - \tfrac{3}{8}c_0 P^2 + c_2 = 0 \tag{3.10}$$

$$a_2 a_{2,xx} - \tfrac{1}{2}a_{2,x}^2 - 2Pa_2^2 + c_3 = 0. \tag{3.11}$$

Denoting parameters $\omega = \alpha c_0/4$, $c_1 = \beta c_0$, $c_3 = \delta$, $c_0 = 8/\nu$ and assuming $c_2 = 0$ (one can change variable x), $P(x) \equiv y(x)$, $a_2(x) \equiv u(x)$, we have the following system of equations from equations (3.10) and (3.11):

$$y_{xx} - 3y^2 + \beta y - \alpha x + \nu u = 0 \tag{3.12}$$

$$uu_{xx} - \tfrac{1}{2}u_x^2 - 2yu^2 + \delta = 0. \tag{3.13}$$

This system of equations can be considered as the Bäcklund transformations between two new fourth-order ODEs for $y(x)$ and $u(x)$. The equation for $u(x)$ can be written in the form

$$u_{xxxx} - 3\frac{u_x u_{xxx}}{u} - \frac{7}{2}\frac{u_{xx}^2}{u} + \frac{17}{2}\frac{u_x^2 u_{xx}}{u^2} - \frac{27}{8}\frac{u_x^4}{u^3} + \left(\beta - \frac{5\delta}{u^2}\right)u_{xx}$$
$$- \frac{1}{2}\left(\frac{\beta}{u} - \frac{15\delta}{u^3}\right)u_x^2 + 2\nu u^2 - 2\alpha x u + \frac{\beta\delta}{u} - \frac{3\delta^2}{2u^3} = 0. \tag{3.14}$$

The equation for $y(x)$ takes the form

$$(\beta y - \alpha x - 3y^2 + y_{xx})y_{xxxx} - \tfrac{1}{2}y_{xxx}^2 + (\alpha - \beta y_x + 6yy_x)y_{xxx}$$
$$+ (\beta - 8y)y_{xx}^2 + (\beta^2 y - 13\beta y^2 - 6y_x^2 + 30y^3 + 10\alpha xy - \alpha x\beta)y_{xx}$$
$$+ (6\alpha x - \tfrac{1}{2}\beta^2)y_x^2 + \alpha(\beta - 6y)y_x - 18y^5 + 12\beta y^4$$
$$- 2(6\alpha x + \beta^2)y^3 + 4\beta\alpha xy^2 - 2\alpha^2x^2y - \tfrac{1}{2}\alpha^2 + \delta\nu^2 = 0. \tag{3.15}$$

From the system of equations (3.12) and (3.13) we can see that equation (3.15) has the special solutions in the form of the general solution of the first Painlevé equation. Therefore, the general solutions of equations (3.14) and (3.15) are also transcendental functions with respect to constants of integration.

Note that equation (3.14) can be written in another form (maybe more convenient) if we use the variable $u \equiv \exp(v(x))$.

In this case we have the fourth-order ODE of the form

$$\frac{d^2}{dx^2}(v_{xx} - \tfrac{1}{2}v_x^2) + (v_{xx} - \tfrac{1}{2}v_x^2)^2 + \beta(v_{xx} + \tfrac{1}{2}v_x^2)$$
$$- 5\delta(v_{xx} - \tfrac{1}{2}v_x^2 + \tfrac{1}{2}\beta)e^{-2v} + 2\nu e^v - \tfrac{1}{2}\delta^2 e^{-4v} - 2\alpha x = 0. \tag{3.16}$$

Equation (3.16) at $\beta = \delta = \nu = 0$ has the special solution that is expressed via the general solution of the first Painlevé equation in the form

$$S_{xx} + S^2 - 2\alpha x = 0 \tag{3.17}$$

where

$$S = v_{xx} - \tfrac{1}{2}v_x^2. \tag{3.18}$$

Assuming $v = -\ln\varphi$ the last equation can be written as the linear equation in the form

$$\varphi_{xx} + S\varphi = 0. \tag{3.19}$$

We feel we need to study equations (3.14)–(3.16) in more detail in future work.

The isomonodromic linear problem for equation (3.14) can be obtained on the basis of the system of equations (2.6). It takes the form

$$\Psi_{xx} = (P_2 - \lambda)\Psi \qquad 2\alpha\lambda\Psi_\lambda = 2A_2\Psi_x - A_{2,x}\Psi \tag{3.20}$$

where $P_2(x)$ and $A_2(x,\lambda)$ are expressed by the following formulae:

$$P_2(x) = \frac{u_{xx}}{2u} - \frac{u_x^2}{4u^2} + \frac{\delta}{2u^2} \tag{3.21}$$

$$A_2(x,\lambda) = \frac{1}{2}\nu u + \left(\frac{u_x x}{u} - \frac{u_x^2}{2u^2} + \frac{\delta}{u^2} - \beta\right)\lambda + 4\lambda^2. \tag{3.22}$$

The system of equations (3.20) is the key to solving the Cauchy problem for equation (3.14).

3.3. Case

$\omega(\lambda) = 2\omega\lambda^2$. Using equations (2.3), (3.1) and (2.1) again we have

$$a_0 = c_0 \tag{3.23}$$

$$a_1 = \tfrac{1}{2}c_0 P + \tfrac{1}{2}\omega x - \tfrac{1}{4}c_1 \tag{3.24}$$

$$4a_{2,x} - 3c_0 P P_x - 2P\omega + \tfrac{1}{2}c_0 P_{xxx} - \omega x P_x + \tfrac{1}{2}c_1 P_x = 0 \tag{3.25}$$

$$a_{2,xxx} - 2P_x a_2 - 4P a_{2,x} = 0. \tag{3.26}$$

The last equation can be integrated. It takes the form

$$a_2 a_{2,xx} - \tfrac{1}{2}a_{2,x}^2 - 2Pa_2^2 + c_3 = 0. \tag{3.27}$$

From equation (3.27) one can obtain $P(x)$. Substituting this dependence into equation (3.25) we can integrate this equation. If we denote $a_2(x) \equiv y(x)$, $\omega = \alpha c_0$, $c_3 = \delta$, $c_2 = \mu$, $c_1 = \beta c_0$, $c_0 = 8/\nu$ (we take the parameter β as zero because one can change variable x)we will obtain the following fourth-order ODE:

$$y_{xxxx} - \frac{4y_x y_{xxx}}{y} - \frac{3y_{xx}^2}{y} + \frac{21}{2}\frac{y_x^2 y_{xx}}{y^2} - \frac{9}{2}\frac{y_x^4}{y^3} - \left(2\alpha x + \frac{5\delta}{y^2}\right)y_{xx}$$
$$+ 2\left(\frac{\alpha x}{y} + \frac{5\delta}{y^3}\right)y_x^2 - 2\alpha y_x + \nu y^2 + \mu - \frac{4\alpha\delta x}{y} - \frac{2\delta^2}{y^3} = 0. \tag{3.28}$$

This equation can be presented in a more convenient form if we use variable $u \equiv \exp(v(x))$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(v_{xxx} - \frac{1}{2}v_x^3\right) - 2\alpha\frac{\mathrm{d}}{\mathrm{d}x}(xv_x) - 5\delta\left(v_{xx} - v_x^2 + \frac{4}{5}\alpha x\right)\mathrm{e}^{-2v} + \nu\mathrm{e}^{v} + \mu\mathrm{e}^{-v} - 2\delta^2\mathrm{e}^{-4v} = 0. \tag{3.29}$$

At $\delta = \mu = \nu = 0$ equation (3.29) takes the form of the second Painlevé equation if we use at $w = v_x$. This form is

$$w_{xx} - \tfrac{1}{2}w^3 - 2\alpha w x + c_5 = 0. \tag{3.30}$$

Equation (3.29) is a generalization of the equation which was found in a recent work [21]. Equation (3.29) was shown to have the special case of the third Painlevé equation if we do not take into account the leading members of this equation. The general solution of equation (3.29) is a transcendental function with respect to constants of integration. This one passes the Painlevé test.

The isomonodromic linear problem for equation (3.28) takes the form

$$\Psi_{xx} = (P_3 - \lambda)\Psi \qquad 8\alpha\lambda^2\Psi_\lambda = 2A_3\Psi_x - A_{3,x}\Psi \tag{3.31}$$

where $P_3(x)$ and $A_3(x, \lambda)$ are expressed by the following formulae:

$$P_3(x) = \frac{y_{xx}}{2y} - \frac{y_x^2}{4y^2} + \frac{\delta}{2y^2} \tag{3.32}$$

$$A_3(x, \lambda) = \frac{1}{2}\nu y + \left(\frac{y_{xx}}{y} - \frac{y_x^2}{2y^2} + \frac{\delta}{y^2} - \beta + 2\alpha x\right)\lambda + 4\lambda^2. \tag{3.33}$$

The general solution of equation (2.19) can be found by the inverse monodromy transform.

3.4. Case

$\omega(\lambda) = 2\omega\lambda^3$. Taking into account equations (2.3) and (3.1) we obtain from equation (2.1)

$$a_0(x) = \tfrac{1}{2}\omega x + c_0 \tag{3.34}$$

$$4a_{1,x} - \omega x P_x - 2c_0 P_x - 2\omega P = 0 \tag{3.35}$$

$$a_{1,xxx} + 4a_{2,x} - 4Pa_{1,x} - 2Pa_1 = 0 \tag{3.36}$$

$$a_2 a_{2,xx} - \tfrac{1}{2}a_{2,x}^2 + c_3 - 2Pa_2^2 = 0. \tag{3.37}$$

From equations (3.35) and (3.36) one can find integrals. They take the form

$$c_0 a_{2,xx} + a_1 a_{1,xx} + \tfrac{1}{2}\omega x a_{2,xx} - \tfrac{1}{2}a_{1,x}^2 + 4a_1 a_2 - \tfrac{1}{2}\omega a_{2,x} + c_1 - 2\omega x a_2 P - 4c_0 a_2 P - 2a_1^2 P = 0 \tag{3.38}$$

$$a_2 a_{1,xx} + a_1 a_{2,xx} + 2a_2^2 - a_{1,x} a_{2,x} + c_2 - 4a_1 a_2 P = 0. \tag{3.39}$$

Substituting $P(x)$ from equation (3.37) into (3.38), (3.39) and denoting $a_1(x) = y(x), a_2(x) = u(x), c_0 = \nu, \omega = \alpha, c_1 = \beta, c_2 = \mu, c_3 = \delta, x' = x + 2\nu/\alpha$, we find the following system of equations:

$$2u^3 + y_{xx}u^2 + (\mu - yu_{xx} - y_x u_x)u + yu_x^2 - 2\delta y = 0 \tag{3.40}$$

$$4u^3 y + (yy_{xx} - \tfrac{1}{2}\alpha x u_{xx} - \tfrac{1}{2}y_x^2 - \tfrac{1}{2}\alpha u_x + \beta)u^2 + (\tfrac{1}{2}\alpha x u_x^2 - \alpha x\delta - y^2 u_{xx})u - \delta y^2 + \tfrac{1}{2}y^2 u_x^2 = 0. \tag{3.41}$$

The Cauchy problem for the latter system of equations can also be solved by the inverse monodromy transform.

4. Fourth-order equations with potential (2.4)

Now consider fourth-order ODEs that can be found from equation (2.1) taking into account potential (2.4) and $A(x, \lambda)$ in the form of equation (3.1).

4.1. Case

$\omega(\lambda) = 2\omega = \text{const}$. Substitutions of equations (2.4) and (3.1) into (2.1) gives the following set of equations:

$$a_0 = c_0 \tag{4.1}$$

$$a_1 = c_0 y + c_1 \tag{4.2}$$

$$a_2 = \tfrac{3}{2}c_0 y^2 - \tfrac{1}{2}c_0 P + c_1 y + \tfrac{1}{4}c_2 \tag{4.3}$$

$$c_0 y_{xx} + 10c_0 y^3 + 6c_1 y^2 + c_2 y + c_3 - 2c_1 P - 6c_0 y P + 4\omega x = 0 \tag{4.4}$$

$$a_{2,xxx} - 4Pa_2 - 4\omega y - 2P_x a_2 = 0. \tag{4.5}$$

Here c_0, c_1, c_2, c_3 and later c_4 are constants of integration again. Substituting $P(x)$ from equation (4.4) into equations (4.3), (4.5) and performing the change of variables and parameters $y(x) = \frac{u(x)}{2} - \frac{c_1}{3c_0}, \omega = \frac{\alpha c_0}{8}, c_1 = \frac{\beta}{2}c_0, c_2 = \delta c_0,\ c_3 = \mu c_0, c_4 = \chi, \alpha x' = \alpha x - \frac{\beta\delta}{3} + \frac{2\beta^3}{27} + \mu, \beta' = (\beta^2 - 3\delta)^{1/2}$ (the primes of variables are omitted) we obtain, after integration, the following equation:

$$u_{xxxx} - 2\frac{u_x u_{xxx}}{u} - 5u^2 u_{xx} - \frac{5}{2}uu_x^2 - \frac{3}{2}\frac{u_{xx}^2}{u} + 2\frac{u_x^2 u_{xx}}{u^2} - 2\frac{\alpha x u_{xx}}{u} + 2\frac{\alpha x u_x^2}{u^2} - 2\frac{\alpha u_x}{u} + \frac{5}{2}u^5 - \beta^2 u^3 + 2\alpha x u^2 + \chi u - \frac{1}{2}\frac{\alpha^2 x^2}{u} = 0. \tag{4.6}$$

Using variable $u = \exp(v(x))$ we can write equation (4.6) in the form

$$\frac{d^2}{dx^2}(v_{xx} + v_x^2) - \frac{1}{2}(v_{xx} + v_x^2)^2 + \chi + \frac{5}{2}e^{4v} - \left(5v_{xx} + \frac{15}{2}v_x^2 + \beta^2\right)e^{2v} + 2\alpha x e^{v} - 2\alpha e^{-v}\frac{d}{dx}(xv_x) - \frac{1}{2}\alpha^2 x^2 e^{-2v} = 0. \tag{4.7}$$

From equations (4.4) and (4.5) one can see there are special solutions of equations (4.6) and (4.7) which are expressed by means of an elliptic Jacobi function.

Equation (4.6) is invariant under transformations y by $-y$ and α by $-\alpha$. This equation also has the special solution at $\alpha = 0$ and $\chi = \beta^2/3$. It takes the form

$$u(x) = \frac{\beta\sqrt{3}}{3}\tanh\left(\frac{\beta\sqrt{3}}{3}x + \varphi_0\right) \tag{4.8}$$

where φ_0 is a arbitrary constant.

Equation (4.6) can be written as the isomonodromic linear problem that corresponds to the compatibility condition (2.5). For equation (4.6) this problem takes the form

$$6\Psi_{xx} = 2(3\lambda - 3u + \beta)\Psi_x - Q_1(x)\Psi \qquad 18\alpha\Psi_\lambda = 6C_1(x,\lambda)\Psi_x + D_1(x,\lambda)\Psi \tag{4.9}$$

where $Q_1(x)$, $C_1(x,\lambda)$ and $D_1(x,\lambda)$ are expressed via the following formulae:

$$Q_1(x) = 3u_x + \frac{\beta^2}{3} - u^2 - \frac{1}{u}(u_{xx} + \alpha x) \tag{4.10}$$

$$C_1(x) = 2u^2 + \beta u - \frac{\beta^2}{3} - \frac{1}{u}(u_{xx} + \alpha x) + (3u + 2\beta)\lambda + 3\lambda^2 \tag{4.11}$$

$$D_1(x) = \beta u^2 + \frac{\beta^3}{3} + 6\alpha x + 6u^3 - 3u_{xx} - 3\beta u_x - 12uu_x - 2\beta^2 u + \frac{1}{u}(3u_{xxx} + 3\alpha + \beta u_{xx} + \alpha\beta x) - \frac{3u_x}{u^2}(u_{xx} + \alpha x) + \left(3u^2 - \beta^2 - 9u_x + \frac{3\alpha x + 3u_{xx}}{u}\right)\lambda - 9\beta\lambda^2 - 9\lambda^3. \tag{4.12}$$

The linear problem (4.9) can be used for solving equation (4.6) at all initial dates by the inverse monodromy transform.

4.2. Case

$\omega(\lambda) = 2\omega\lambda$. Using the above-mentioned approach we have $a_0(x) = c_0$, $a_1(x) = c_0 y(x) + c_1$ and equations in the form

$$a_2(x) = \omega x + \frac{3}{2}c_0 y^2 - \frac{c_0}{2}P + c_1 y + c_2 \tag{4.13}$$

$$c_0 y_{xx} - 6c_0 P y - 2c_1 P + 4\omega y x + 10 c_0 y^3 + 6c_1 y^2 + 4c_2 y + c_3 = 0 \tag{4.14}$$

$$a_2 a_{2,xx} - \tfrac{1}{2}a_{2,x}^2 - 2Pa_2^2 + c_4 = 0. \tag{4.15}$$

One can find $P(x)$ from equation (4.14). Substituting this expression into (4.13), (4.15) and using parameters $\omega = \alpha c_0$, $c_1 = \beta c_0$, $c_2 = c_0\nu$, $c_3 = c_0(\mu - \frac{2\beta^3}{27})$, $c_4 = \delta c_0^2/16$, we have after the change of variables

$$x' = x + \frac{\nu}{\alpha} - \frac{1}{6\beta^2\alpha}, \qquad a_2(x) = u(x)/4, \qquad y' = y - \frac{\beta}{3} \tag{4.16}$$

(primes are omitted) the system of equations

$$(3u - 8\alpha x)y + y_{xx} - 8y^3 - 4\beta y^2 - \tfrac{4}{3}\alpha x\beta + \mu = 0 \tag{4.17}$$

$$(-y_{xx} + 4\beta y^2 - 10y^3 - 4\alpha x y + \tfrac{4}{3}\alpha x\beta - \mu)u^2 + 3yuu_{xx} + (3\delta - \tfrac{3}{2}u_x^2)y = 0. \tag{4.18}$$

This system can be considered as the second member of the fourth Painlevé hierarchy. Recently this one was obtained as reduction of the Hirota–Satsuma system of equations in [23] where the Bäcklund transformations and special solutions were found.

The isomonodromic linear problem corresponding to equations (4.17) and (4.18) takes the form

$$9\Psi_{xx} = 3(3\lambda - 6y + 2\beta)\Psi_x - Q_2(x)\Psi \qquad 4\alpha\lambda\Psi_\lambda = C_2(x,\lambda)\Psi_x + D_2(x,\lambda)\Psi \tag{4.19}$$

where

$$Q_2(x) = \beta^2 - 6y^2 + 9y_x - 6\alpha x + \frac{1}{2y}(4\alpha\beta x - 3\mu - 3y_{xx}) \tag{4.20}$$

$$C_2(x,\lambda) = u(x) + 2(y + \tfrac{2}{3}\beta)\lambda + \lambda^2 \tag{4.21}$$

$$D_2(x,\lambda) = uy - \tfrac{1}{2}u_x - \tfrac{1}{3}\beta u - \beta\lambda^2 - \tfrac{1}{2}\lambda^3 + (2\alpha x - y_x + 2y^2 - \tfrac{1}{2}u + \tfrac{2}{3}\beta y - \tfrac{4}{9}\beta^2)\lambda. \tag{4.22}$$

The isomonodromic linear problem (4.19) will be used for solving the Cauchy problem of equations (4.17) and (4.18).

4.3. Case

$\omega(\lambda) = 2\omega\lambda^2$. Substituting equations (3.1) and (2.4) into (2.1), we have $a_0(x) = c_0$ and the following equations:

$$a_1(x) = c_0 y + \omega x + c_1 \tag{4.23}$$

$$a_2(x) = \omega y x + \tfrac{3}{2}c_0 y^2 - \tfrac{1}{2}c_0 P(x) + c_1 y + c_2 \tag{4.24}$$

$$-4c_0 P y_x - 4P\omega + 8y a_{2,x} + 4y_x a_2 + c_0 y_{xxx} - 2c_0 P_x y - 2\omega x P_x - 2c_1 P_x = 0 \tag{4.25}$$

$$a_2 a_{2,xx} - \tfrac{1}{2}a_{2,x}^2 - 2Pa_2^2 + c_4 = 0. \tag{4.26}$$

Substituting $P(x)$ from equation (4.26) into (4.25) and multiplying the expression found by $a_2(x)$ we obtain, after integration, some equations. We have used further parameters and variables: $\omega = \alpha c_0, c_1 = \beta c_0, c_2 = \mu c_0/4, c_3 = \nu c_0^2, c_4 = \delta c_0^2, a_2(x) = c_0 u(x), y' = y - \alpha x$ (the prime of y' is omitted). As a result we have the following system of equations obtained from equations (4.26) and (4.25):

$$u^2 y_{xx} - u(yu_{xx} + y_x u_x) + y(u_x^2 - 2\delta) + 4u^3(y - \alpha x) + \nu u = 0 \tag{4.27}$$

$$uu_{xx} - \tfrac{1}{2}u_x^2 - 2u^2(3y^2 + \alpha^2 x^2) + u^2(8\alpha xy - \mu) + 4u^3 + \delta = 0. \tag{4.28}$$

The isomonodromic linear problem for the system of equations (4.27) and (4.28) can be presented in the form

$$\Psi_{xx} = (\lambda - 2y + 2\alpha x)\Psi_x - Q_3(x)\Psi \qquad 2\alpha\lambda^2\Psi_\lambda = C_3(x,\lambda)\Psi_x + D_3(x,\lambda)\Psi \tag{4.29}$$

where

$$Q_3(x) = y_x - \alpha + (y - \alpha x)^2 - \frac{u_{xx}}{2u} + \frac{1}{4u^2}(u_x^2 - 2\delta) \tag{4.30}$$

$$C_3(x,\lambda) = 4u + 2y\lambda + \lambda^2 \tag{4.31}$$

$$D_3(x,\lambda) = 4u(y - \alpha x) - 2u_x + (2y^2 - 2u - y_x - 2\alpha xy)\lambda - \tfrac{1}{2}\lambda^3. \tag{4.32}$$

The isomonodromic linear problems (4.29) can be used to solve the Cauchy problem for equations (4.27), (4.28) by the inverse monodromy transform.

4.4. Case

$\omega(\lambda) = 2\omega\lambda^3$. Substituting equations (2.4) and (3.1) into (2.1) we have

$$a_0(x) = \omega x + c_0 \tag{4.33}$$

$$a_1(x) = \omega xy + c_0 y - c_1 \tag{4.34}$$

$$12(\omega x + c_0)yy_x - 2(\omega x + c_0)P_x + 8\omega y^2 - 4\omega P - 4c_1 y_x - 4a_{2,x} = 0 \tag{4.35}$$

$$\begin{aligned}(\omega x + c_0)y_{xxx} &- 2y(\omega x + c_0)P_x - 4(\omega x + c_0)Py_x \\ &+ 3\omega y_{xx} + 8ya_{2,x} - 4\omega yP + 4a_2 y_x + 2c_1 P_x = 0\end{aligned} \tag{4.36}$$

$$a_2 a_{2,xx} - \tfrac{1}{2}a_{2,x}^2 + c_3 - 2Pa_2^2 = 0. \tag{4.37}$$

Using $P(x)$ from equation (4.37) and denoting $a_2(x) = u(x), c_0 = \nu, \omega = \alpha, c_1 = \beta, c_2 = \mu, c_3 = \delta, c_4 = \varkappa, x' = x + \nu/\alpha$, we obtain from equations (4.35) and (4.36) after integration

$$\begin{aligned}\alpha xu^2 y_{xx} &+ (\beta - \alpha xy)(uu_{xx} - u_x^2) - \alpha u(y + xy_x)u_x \\ &+ 2\alpha y_x u^2 + 4yu^3 - 2\alpha x\varkappa y + \delta u + 2\beta\varkappa = 0\end{aligned} \tag{4.38}$$

$$\begin{aligned}\alpha x(\alpha xy - \beta)u^2 y_{xx} &- ((\alpha xy - \beta)^2 u + \alpha xu^2)u_{xx} + (\alpha xu + \tfrac{1}{2}(\alpha xy - \beta)^2)u_x^2 - \tfrac{1}{2}\alpha^2 x^2 y_x^2 u^2 \\ &+ \alpha(\alpha xy - 2\beta)u^2 y_x - \alpha u^2 u_x - 2u^4 + 8(\alpha xy - \beta)yu^3 \\ &+ (\mu - \tfrac{1}{2}\alpha^2 y^2)u^2 - 2\alpha x\varkappa u - \alpha^2 x^2 \varkappa y^2 + 2\alpha x\beta\varkappa y - \varkappa\beta^2 = 0.\end{aligned} \tag{4.39}$$

The Cauchy problem for the system of equations (4.38) and (4.39) are solved by the inverse monodromy transform.

5. The Painlevé test for equations (3.14), (3.28), (4.6) and the system of equations (4.27), (4.28)

The Painlevé test is known to be a powerful method for investigating the integrability of differential equations [1,24]. This approach allows one to obtain the necessary conditions for the absence of movable critical singularities in the general solution of a differential equation.

The Cauchy problem for the equations presented above can be solved by the inverse monodromy transform. We expect all these equations to pass the Painlevé test. However, the Painlevé test also gives the representation of the general solution near a singular point and the formulae found are useful to prove the local convergence of the general solution and other interesting features of solution [25].

We are going to apply the Painlevé test to equations (3.14), (3.28), (4.6) and to the system of equations (4.27) and (4.28) in this section using the perturbative Painlevé approach presented in [1, 24]. The essence of this method can be presented in several ways. Let the ODE

$$E(y, y_x, \ldots, x) = 0 \tag{5.1}$$

be given, where we keep in mind equations (3.14), (3.28), and (4.6). It is convenient to use the perturbative method taking into account the following three steps.

In the first place we look for all possible families of solutions of equation (5.1) assuming

$$y = y_0 z^p, \qquad z = x - x_0 \tag{5.2}$$

where p is the order of the singularity, y_0 is a coefficient and x_0 is a movable singularity. Substituting (5.2) into leading members of equation (5.1), we get several families of solutions with values (p, y_0). To continue the investigation of the equation we have to obtain all integer values of p in this step.

To study the second necessary condition, we look for the Fuchs indices. At this step we assume

$$y = y_0 z^p + y_i z^{p+j} \tag{5.3}$$

for every family of solutions and substitute expression (5.3) into the leading members of equation (5.1) again. Equating expressions at y_j in this step we obtain the integer Fuchs indices j_r, $(r = 1, \ldots, n)$, where n is the order of equation (5.1).

The third necessary condition corresponds to checking the existence of the Laurent series for the general solution of equation (5.1). At this step, the general solution of equation (5.1) is searched for in the form

$$y = \sum_{k=0}^{\infty} \varepsilon^k y^{(k)}. \tag{5.4}$$

Equation (5.1), in this case, takes the form

$$E(z, y) = \sum_{k=0}^{\infty} \varepsilon^k E^{(k)} = 0. \tag{5.5}$$

We have [1, 24] from equation (5.5)

$$k = 0: \qquad E^{(0)} \equiv E'(z, y^{(0)}) = 0 \tag{5.6}$$

$$k = 1: \qquad E^{(1)} \equiv E'(z, y^{(0)}) y^{(1)} = 0 \tag{5.7}$$

$$k \geqslant 2: \qquad E^{(k)} \equiv E'(z, y^{(0)}) y^{(k)} + R^{(k)}(z, y^{(0)}, \ldots, y^{(k-1)}) = 0 \tag{5.8}$$

where E' is the Gateaux derivative of equation (5.1) and $R^{(k)}$ stands for the contribution of previous members of the expressions. The components of the solution $y^{(k)}$ are looked up in terms of the Laurent series

$$y^{(k)} = \sum_{j=k\rho}^{\infty} y_j^{(k)} z^{j-p} \tag{5.9}$$

where ρ is the least negative Fuchs index. Solution (5.4) must have n arbitrary constants to pass the Painlevé test. The coefficients $y_j^{(k)}$ are found after the substitution of representation (5.9) into equations (5.6)–(5.8) and equating the obtained coefficients $E_j^{(k)}$ to zero. The absence of movable critical points corresponds to n arbitrary constants in solution (5.4). Arbitrary coefficients $y_r^{(k)}$ ($r = 1, \ldots, n$); $r \leqslant n$; r is the Fuchs index) are introduced at $k = 0$ for $r \geqslant 0$ and $k \geqslant 1$ for $r \leqslant -1$.

Let us apply this algorithm to the investigation of equations (3.14), (3.28), (4.6) and the system of equations (4.27), (4.28) in the Painlevé test.

5.1. Test for equation (3.14).

To study equation (3.14) let us write this one in the form

$$y^3 y_{xxxx} - 3y^2 y_x y_{xxx} - \frac{7}{2} y^2 y_{xx} + \frac{17}{2} y y_x^2 y_{xx} - \frac{27}{8} y_x^4 + 2\nu y^5 + \left(\beta - \frac{5\delta^2}{2y^2}\right) y^3 y_{xx}$$
$$- \frac{1}{2}\left(\beta - \frac{15\delta^2}{2y^2}\right) y^2 y_x^2 - 2\alpha x y^4 + \frac{1}{2}\beta\delta^2 y^2 - \frac{3}{8}\delta^4 = 0. \tag{5.10}$$

Substituting (5.2) into the leading members of equation (5.10) we have five families of solutions with values $(p, y_0) = (-4, 72/\nu)$, $(p, y_0) = (1, \delta)$, $(p, y_0) = (1, -\delta)$, $(p, y_0) = (1, \delta/3)$ and $(p, y_0) = (1, -\delta/3)$.

Substituting (5.3) where $y_0 = 72/\nu$ and $p = -4$ into the leading members of equation (5.10) again, we find equation for the Fuchs indices of the form

$$j^4 - 10j^3 - 5j^2 - 150j + 144 = 0. \tag{5.11}$$

The solution of equation (5.11) takes the form $j_1 = -1$, $j_2 = -3$, $j_3 = 6$ and $j_4 = 8$. Therefore the first family of solutions has two positive Fuchs indices and two negative ones.

Substituting (5.3) into the leading members of equation (5.10) for the second and third families gives the equation

$$j^4 - 5j^3 + 5j^2 + 5j - 6 = 0. \tag{5.12}$$

The solutions of equation (5.12) take the form $j_1 = -1$, $j_2 = 1$, $j_3 = 2$ and $j_4 = 3$.

In the case of the fourth and fifth families of solutions we obtain the equation for the Fuchs indices in the form

$$j^4 - 5j^3 - 15j^2 + 45j - 54 = 0 \tag{5.13}$$

that has solutions $j_1 = -1$, $j_2 = -3$, $j_3 = 3$ and $j_4 = 6$.

One can see that the first and second necessary conditions for passing the Painlevé test for equation (5.10) are satisfied.

The solution $y^{(0)}$corresponding to the first family takes the form

$$y^{(0)} = \frac{72}{\nu(x-x_0)^4} - \frac{12\beta}{5\nu(x-x_0)^2} + \frac{\beta^2}{350\nu} + \frac{18\alpha x_0}{35\nu}$$
$$+ \frac{\alpha}{\nu}(x-x_0) + y_6(x-x_0)^2 - \frac{2\alpha\beta}{75\nu}(x-x_0)^3 + y_8(x-x_0)^4 + \cdots. \tag{5.14}$$

This solution has three arbitrary constants x_0, y_6 and y_8. However, substituting

$$y(x) = y^{(0)} + \varepsilon y^{(1)} + \varepsilon^2 y^{(2)} + \varepsilon^3 y^{(3)} \tag{5.15}$$

into equation (5.10) and using (5.14) and (5.9) we find that $y(x)$ has four arbitrary constants.

Solution $y^{(0)}$ corresponding to the second and third families can be written in the form

$$y^{(0)} = \pm\delta(x-x_0) + y_1(x-x_0)^2 + y_2(x-x_0)^3 + y_3(x-x_0)^4$$
$$+ \left(\pm\frac{\delta\alpha x_0}{15} - \frac{\beta y_2}{10} \pm \frac{2y_1y_3}{3\delta} \pm \frac{3y_2^2}{4\delta} - \frac{y_1^2 y_2}{3\delta^2}\right)(x-x_0)^5 + \cdots. \tag{5.16}$$

The latter solutions already have four arbitrary constants x_0, y_1, y_2 and y_3 that corresponding to the positive Fuchs indices and $j_1 = -1$.

Solution $y^{(0)}$ for the fourth and fifth families of solutions have the form

$$y^{(0)} = \pm\frac{1}{3}\delta(x - x_0) \mp \frac{1}{45}\beta\delta(x - x_0)^3 + y_3(x - x_0)^4 \mp \left(\frac{\delta\beta^2}{1260} + \frac{\delta\alpha x_0}{105}\right)(x - x_0)^5 + \left(\frac{\beta y_3}{30} + \frac{\nu\delta^2}{432} \mp \frac{\alpha\delta}{144}\right)(x - x_0)^6 + y_6(x - x_0)^7 + \cdots. \quad (5.17)$$

Solution (5.17) has three arbitrary constants x_0, y_3 and y_6; but using solutions (5.15), (5.17) and (5.9) one can find from equations (5.6)–(5.8) that $y(x)$ has four arbitrary constants for $1 \leqslant k \leqslant 4$. We have found that equation (5.10) passes the Painlevé test.

5.2. Test for equation (3.28).

Now consider the application of the Painlevé test to equation (3.28). We will use this equation in the form

$$y^3 y_{xxxx} - 4y^2 y_x y_{xxx} + \frac{21}{2} y y_x^2 y_{xx} - 3y^2 y_{xx}^2 - \frac{9}{2} y_x^4 - \left(2\alpha x + \frac{5\delta^2}{2y^2}\right) y^3 y_{xx} + 2y^2\left(\alpha x + \frac{5\delta^2}{2y^2}\right) y_x^2 - 2\alpha y_x y^3 + \nu y^5 + \mu y^3 - 2\alpha\delta^2 x y^2 - \frac{1}{2}\delta^4 = 0. \quad (5.18)$$

Substituting (5.2) into leading members of equation (5.18) we get five families of solutions with values $(p, y_0) = (-4, 72/\nu)$, $(p, y_0) = (1, \delta)$, $(p, y_0) = (1, -\delta)$, $(p, y_0) = (1, \delta/3)$ and $(p, y_0) = (1, -\delta/3)$.

Substitution of (5.3) into leading members of equation (5.18) gives the equation, which corresponds to the first family of solution, in the form

$$j^4 - 6j^3 - 13j^2 + 66j + 72 = 0. \quad (5.19)$$

The Fuchs indices for the first family of solutions are found from equation (5.19). They take the form $j_1 = -1$, $j_2 = -3$, $j_3 = 4$, and $j_4 = 6$.

For the second and third families of solution we have, after substitution of (5.3) into leading members of equation (5.18), the following equation:

$$j^4 - 6j^3 + 7j^2 + 6j - 8 = 0. \quad (5.20)$$

Solutions of equation (5.20) take the form $j_1 = -1$, $j_2 = 1$, $j_3 = 2$, $j_4 = 4$.

Substitution of (5.3) into the leading members of equation (5.18) for the fourth and fifth families of solutions leads to equation (5.19). Therefore, the Fuchs indices coincide in this case with the Fuchs indices of the first family of solutions.

Solution $y^{(0)}$ for the first family can be written in the form

$$y^{(0)} = \frac{72}{\nu(x - x_0)^4} + \frac{24\alpha x_0}{5\nu(x - x_0)^2} + a_4 - \frac{4\alpha^2 x_0}{5\nu}(x - x_0) + a_6(x - x_0)^2 + \left(\frac{2}{15}\alpha a_4 - \frac{32\alpha^3 x_0^2}{375\nu}\right)(x - x_0)^3 + \cdots. \quad (5.21)$$

One can see from (5.21) that we have only three arbitrary constants. However, using the solution in the form (5.15), we find four arbitrary constants in $y(x)$ after solving equations such as (5.6)–(5.8) at $k \leqslant 4$.

The solution $y^{(0)}$ for the second and third families of solutions can be presented in the form

$$y^{(0)} = \pm\delta(x-x_0) + b_1(x-x_0)^2 + b_2(x-x_0)^3 + \left(\frac{\alpha b_1 x_0}{2} + \frac{\mu}{8} \pm \frac{5b_1b_2}{4\delta} \mp \frac{\alpha\delta}{4}\right)(x-x_0)^4 + b_4(x-x_0)^5 + \cdots. \quad (5.22)$$

We can see from solution (5.22) that there are four arbitrary constants x_0, b_0, b_2 and b_4 in $y^{(0)}$.

We obtain $y^{(0)}$ for the fourth and fifth families of solutions in the form

$$y^{(0)} = \pm\frac{\delta}{3}(x-x_0) \pm \frac{2\alpha x_0\delta}{45}(x-x_0)^3 \pm \frac{1}{12}\left(\alpha\delta \mp \frac{\mu}{6}\right)(x-x_0)^4 + w_4(x-x_0)^5 + \left(\frac{\delta^2\nu}{432} \pm \frac{\delta\alpha^2 x_0}{270} + \frac{\alpha x_0\mu}{1080}\right)(x-x_0)^6 + w_6(x-x_0)^7 + \cdots. \quad (5.23)$$

We have three arbitrary constants x_0, w_4 and w_6 in solution (5.23). However, using perturbations for $y(x)$ in the form (5.15), we find four arbitrary constants for the fourth and five families of solutions.

Taking into account all three steps for every family of solutions of equation (5.18) we make the conclusion that equation (5.18) passes the Painlevé test.

5.3. Test for equation (4.6).

Let us examine equation (4.6) in the Painlevé test. We take this equation in the form

$$y^2 y_{xxxx} + 2y_x^2 y_{xx} - \tfrac{3}{2} y y_{xx}^2 + \tfrac{5}{2} y^7 - 5y^4 y_{xx} - \tfrac{5}{2} y^3 y_x^2 - 2y y_x y_{xxx} + 2\alpha x(y_x^2 - y y_{xx} + y^4) - \tfrac{1}{2}\alpha^2 x^2 y - 2\alpha y_x y - \beta^2 y^5 + \chi y^3 = 0. \quad (5.24)$$

Substituting (5.2) into the leading members of equation (5.24) we find four families of solutions with values $(p, y_0) = (-1, 1), (p, y_0) = (-1, -1), (p, y_0) = (-1, 2)$ and $(p, y_0) = (-1, -2)$. The equation for the Fuchs indices of the first and second families of solutions takes the form

$$j^4 - 8j^3 + 14j^2 + 8j - 15 = 0. \quad (5.25)$$

This equation has solutions $j_1 = -1$, $j_2 = 1$, $j_3 = 3$ and $j_4 = 5$.

The equation for the Fuchs indices corresponding to the third and fourth families of solutions can be presented in the form

$$j^4 - 8j^3 - j^2 + 68j + 60 = 0. \quad (5.26)$$

This equation has solutions $j_1 = -1$, $j_2 = -2$, $j_3 = 5$ and $j_4 = 6$.

Solutions $y^{(0)}$ for the first and second families of solutions take the form

$$y^{(0)} = \pm\frac{1}{x-x_0} + u_1 \pm \left(\frac{\beta^2}{9} - u_1^2\right)(x-x_0) + u_3(x-x_0)^2 + \left(\frac{2\alpha}{15} \pm \frac{4\alpha x_0 u_1}{15} \pm \frac{4u_1u_3}{3} \pm \frac{\chi}{15} \mp \frac{\beta^4}{162} \pm \frac{2\beta^2 u_1^2}{9} \mp \frac{7u_1^4}{3}\right)(x-x_0)^3 + u^5(x-x_0)^4 + \cdots. \quad (5.27)$$

From equation (5.27) one can see that $y^{(0)}$ has four arbitrary constants x_0, u_1, u_3 and u_5.

Solutions $y^{(0)}$ corresponding to the third and fourth families of solutions can be written in the form

$$y^{(0)} = \pm\frac{2}{x-x_0} \pm \frac{\beta^2}{18}(x-x_0) - \frac{\alpha x_0}{20}(x-x_0)^2 \pm \left(\frac{\beta^4}{648} - \frac{\chi}{30} \mp \frac{2\alpha}{15}\right)(x-x_0)^3 + v_5(x-x_0)^4 + v_6(x-x_0)^5 + \cdots. \quad (5.28)$$

From solution (5.28) one can see that $y^{(0)}$ has three arbitrary constants x_0, v_5 and v_6. However, taking this solution and perturbative solution (5.15) into account, we find by solving similar equations (5.6)–(5.8) that $y(x)$ has four arbitrary constants. We have obtained for equation (5.24) that all three necessary conditions for the Painlevé test are satisfied.

5.4. Test for the system of equations (4.27) and (4.28)

Let us investigate the system of equations (4.27) and (4.28) in the Painlevé test.

Substituting formula (5.2) for y and u (we change $y_0 \to u_0$ and $p \to q$) into leading members of the system of equations (4.27) and (4.28) (where we use $\delta \to \delta^2/2$) we find four families of pairs of solutions with values $(p, y_0) = (-1, 1), (q, u_0) = (-2, 1/2)$; $(p, y_0) = (-1, -1), (q, u_0) = (-2, 1/2)$; $(p, y_0) = (1, \nu/\delta), (q, u_0) = (1, \delta)$; and $(p, y_0) = (1, -\nu/\delta), (q, u_0) = (1, -\delta)$.

Assuming

$$y = y_0 x^p + y_j x^{p+j}, \qquad u = u_0 x^q + u_j x^{q+j} \tag{5.29}$$

in the system of equations (4.27) and (4.28) (where we change $\delta \to \delta^2/2$ for convenience of calculations) and equating linear expressions with respect to y_j and u_j we have the following system of equations for the first and second pair of families of solutions:

$$y_j(j^2 - j) \pm u_j(4 - 4j - 2j^2) = 0 \tag{5.30}$$

$$6y_j \pm u_j(3j - 2 - j^2) = 0. \tag{5.31}$$

The solutions of equations (5.30) and (5.31) at $y_j \neq 0$ and $u_j \neq 0$ take the form $j_1 = -1$, $j_2 = -2$, $j_3 = 3$ and $j_4 = 4$.

Substituting (5.29) into leading members of the system of equations for the third and fourth pair of families of solutions we find the multiple Fuchs indices $j_{1,2} = -1$, $j_{3,4} = 1$.

Solutions $y^{(0)}$ and $u^{(0)}$ for the first and second pair of families of equations can be written in the form

$$y^{(0)} = \pm\frac{1}{x - x_0} + \frac{2}{3}\alpha x_0 + \left(\frac{2}{3}\alpha \pm \frac{1}{9}\alpha^2 x_0^2 \mp \frac{1}{12}\mu\right)(x - x_0) + a_3(x - x_0)^2 + a_4(x - x_0)^3 + \cdots \tag{5.32}$$

$$\begin{aligned} u^{(0)} = {} & \frac{1}{2(x - x_0)^2} \pm \frac{\alpha x_0}{3(x - x_0)} + \frac{\alpha^2 x_0^2}{18} \pm \frac{\alpha}{6} + \frac{\mu}{24} \\ & + \left(\pm\frac{\alpha^3 x_0^3}{27} - \frac{4\alpha^2 x_0}{9} \mp \frac{\alpha x_0 \mu}{36} \pm 3a_3\right)(x - x_0) \\ & + \left(\frac{\alpha^4 x_0^4}{162} \pm a_4 \mp \frac{\alpha^3 x_0^2}{27} + \frac{\mu^2}{228} - \frac{\alpha^2 x_0 \mu}{108} + \frac{\alpha x_0 a_3}{3} - \frac{\alpha^2}{18}\right)(x - x_0)^3 + \cdots. \end{aligned} \tag{5.33}$$

The solutions (5.32) and (5.33) have three arbitrary constants x_0, a_3 and a_4. However, application of these solutions in the perturbative method leads to the four arbitrary constants.

Solutions $y^{(0)}$ and $u^{(0)}$ for the third and fourth families of solutions can be presented in the form

$$y^{(0)} = \pm\delta(x - x_0) + p_1(x - x_0)^2 \pm \tfrac{1}{3}(\delta\mu + 2\delta\alpha^2 x_0^2)(x - x_0)^3 + \cdots \tag{5.34}$$

$$u^{(0)} = \pm\frac{\nu(x - x_0)}{\delta} + a_1(x - x_0)^2 \pm \frac{1}{3}\left(\frac{\nu\mu}{\delta} + 4\alpha\delta x_0 + \frac{2\nu\alpha^2 x_0^2}{\delta}\right)(x - x_0)^3 + \cdots. \tag{5.35}$$

We have two arbitrary constants in every solution (5.34) and (5.35). However, considering the next step in the perturbative method we obtain four arbitrary constants for $y(x)$ and $u(x)$.

Therefore, the system of equations (4.27) and (4.28) passes the Painlevé test as other equations studied in this section.

6. Conclusion

In this paper we have used the compatibility condition for five of six Painlevé equations to find several new fourth-order ODEs that are analogies of the Painlevé equations. Most of the fourth-order ODEs found are new. The Cauchy problems for these equations can be solved by the inverse monodromy transform and we have given the isomonodromic linear problems for several fourth-order analogies. Special solutions of new ODEs were discussed. As a rule, the special solutions can be found using solutions of the usual Painlevé equations. Consequently, we hope the above-mentioned equations have general solutions in the form of new transcendental functions with respect to constants of integration.

Three new fourth-order ODEs (3.14), (3.28), (4.6) and the system of equations (4.27), (4.28) were studied in the Painlevé test. As expected these equations passed the Painlevé test. The Cauchy problems for these equations can be solved by the inverse monodromy transform taking into account the above-mentioned Lax pairs.

We call the equations presented analogies to the Painlevé equations taking into account that these new ODEs are obtained as the compatibility conditions of the isomonodromic linear problem of the Painlevé equations. We have also considered two types of potentials as it was done in the case of the Painlevé equations. However, we do not think that our fourth-order ODEs are similar to the Painlevé equations. Much more than that the equations found have special solutions of some Painlevé equations and we think that some fourth-order ODEs can be generalizations of the Painlevé equations. We hope this statement can be proved by means of formula (3.1). One can see that assuming $a_0(x) = 0$ and $a_2(x) \neq 0$ in this formula, we can have one of the Painlevé equations but in the case of $a_0(x) \neq 0$ and $a_2(x) = 0$ we have to obtain another Painlevé equation. Therefore, we think that the fourth-order ODEs found are generalizations of the Painlevé equations.

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