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Double Bäcklund transformations and special integrals for the K_{II} hierarchy

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Abstract

The double Bäcklund Transformations for solutions of the K_{II} hierarchy are given. On the one hand these transformations are found via invariants under Miura transformations for two solutions of the K_{II} hierarchy. On the other hand the double Bäcklund transformations are obtained from the special integrals of this hierarchy taking into account the Painlevé truncation approach. Transformations are applied to look for the rational solutions of the K_{II} hierarchy. © 2000 Published by Elsevier Science B.V.

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1. Introduction

More than one century ago Painlevé and his school began an investigation of the second-order ordinary differential equation (ODE) class. They had two related objectives: to classify the second-order differential equations of a certain form on the basis of their possible solutions singularities, and to identify second-order differential equations defined new functions. As a matter of fact the problem to find new functions (defined as only by solutions of nonlinear ODEs) was formulated by L. Fuchs and A. Poincaré in 1884. Painlevé and his collaborators showed that out of all possible equations of a certain form there are 50 types whose solutions have no movable critical points. This property now called the ODEs Painlevé property.

Furthermore they found that among these 50 equations, 44 are solvable in terms of previously known functions (such as elliptic functions and solutions of linear equations) or are reducible to one of six new nonlinear ODEs. They also showed that there are exactly six second-order ordinary differential equations that define new functions. At the present time these functions are called Painlevé transcendents and equations with general solutions in the form of transcendents are called Painlevé equations. These six Painlevé equations were first discovered from strictly mathematical consideration but they have recently appeared in several physical applications [1].

Current interest in the Painlevé equations stemmed from the observations made by Ablowitz and Segur

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[2] and Ablowitz et al. [3,4] that reductions of partial differential equations of the soliton type give rise to ODEs whose movable singularities are only poles.

In outgoing century a number of scientists have extended Painlevé's first objective and gave a partial classification of the third and fourth-order ordinary differential equations. Painlevé and his collaborators results also led to the problem of finding other new functions could be defined by nonlinear ODEs similar to Painlevé equations.

The further attempt to solve the problem formulated by Poincaré and Fuchs were undertaken in [5] where the P_{I} and P_{II} hierarchies were presented. These hierarchies take the form

$$L^{n+1}[y] = \frac{z}{2} \quad (n=1,2,\dots) \tag{1.1}$$

and

$$\left(\frac{d}{dz} + y\right) L^n\left[y_z - \tfrac{1}{2}y^2\right] - zy - \alpha_n = 0 \tag{1.2}$$

where operator L^n is determined by the Lenard relation using following formulas [6]:

$$\frac{d}{dz} L^{n+1} = L^n_{zzz} + 4yL^n_z + 2y_z L^n, \quad L^0[y] = \tfrac{1}{2},$$
$$L^1[y] = y \tag{1.3}$$

Actually the P_{II} hierarchy (1.2) were considered early in [7–10].

Eqs. (1.1) and (1.2) give the first and second Painlevé equations at $n=1$ but we obtain two fourth-order ordinary differential equations at $n=2$. These equations have a number of remarkable properties like Painlevé equations. Firstly they pass the Painlevé test [11]. Each of them can be written as the compatibility condition of a corresponding linear system. This system allows to solve Eqs. (1.1) and (1.2) using the inverse monodromy transform method [11]. Eq. (1.2) has the Bäcklund transformations, rational solutions and special integrals [7–12]. Solutions of Eqs. (1.1) and (1.2) at $n=2$ were proved to be essentially transcendental functions with respect to constants of integration [13,14].

Recently two other hierarchies were suggested [15]

$$H_n[y] = z, \quad (n=1,2,\dots) \tag{1.4}$$

and

$$\left(\frac{d}{dz} + y\right) H_n\left[y_z - \tfrac{1}{2}y^2\right] - zy - \beta_n = 0 \tag{1.5}$$

where the operator H_n is determined by the following form [16–18]:

$$H_{n+2} = J_1[y]\,\theta_1[y]\,H_n, \quad H_0[y] = 1,$$
$$H_1[y] = y_{zz} + 4y^2 \tag{1.6}$$
$$J_1 = D^3 + 3(yD + Dy) + 2\left(D^2yD^{-1} + D^{-1}yD^2\right) + 8\left(y^2D^{-1} + D^{-1}y^2\right),$$
$$D = \frac{d}{dz}, \quad D^{-1} = \int dz, \tag{1.7}$$
$$\theta_1 = D^3 + 2yD + y_z \tag{1.8}$$

Hierarchies (1.4) and (1.5) were obtained as the similarity reductions from Schwarzian Kaup–Kuperschmidt and modified Kaup–Kuperschmidt hierarchies. It can be shown that the similarity reductions of the Schwarzian Caudrey–Dodd–Gibbon and modified Caudrey–Dodd–Gibbon hierarchies also give hierarchies (1.4) and (1.5).

Taking into account recent 150th anniversary of Sophie Kowalevski and Martin David Kruskal's 75th birth-year let us call hierarchies (1.4) and (1.5) the K_{I} and K_{II} hierarchies.

Eq. (1.4) at $n=1$ leads to the first Painlevé equation but Eq. (1.5) gives the fourth-order ordinary differential equation. Eq. (1.4) at $n=2$ and (1.5) at $n=1$ are shown to be fourth-order ordinary differential equations that pass the Painlevé test and have general solutions in the form of transcendental functions with respect to constant of integration [14].

The aim of this Letter is to present some properties of hierarchy (1.5).

The outline of this Letter is as follows. The double Bäcklund transformations for the solutions of the K_{II} hierarchy are found in Section 2. The application of the Painlevé truncation approach to the K_{II} hierarchy is given in Section 3 where we get singular manifold equations and special integrals for the K_{II} hierarchy. At the same time we obtain the double Bäcklund transformations from the special integrals. Section 4 is devoted to rational solutions of the K_{II} hierarchy. They are found taking into account the double Bäcklund transformations.

2. Double Bäcklund transformations for solutions of the K_{II} hierarchy

Let us consider the K_{II} hierarchy

$$\left(\frac{d}{dz}+y\right)H_n\left[y_z-\tfrac{1}{2}y^2\right]-zy-\beta_n=0$$

$$(n=1,2,\dots) \tag{2.1}$$

We want to look for the Bäcklund transformations of Eq. (2.1). The following theorem can be formulated.

Theorem 1. *Let $y(z,\beta_n)$ and $y(z,2-\beta_n)$ be solutions of Eq. (2.1); then there is an invariant*

$$I_1=y(z,\beta_n)-\frac{\beta_n}{P}=y(z,2-\beta_n)-\frac{2-\beta_n}{P} \tag{2.2}$$

where

$$P=H_n\left[y_z-\tfrac{1}{2}y^2\right]-z \tag{2.3}$$

Proof. Hierarchy (2.1) can be written as the set of equations

$$H_n\left[y_z-\tfrac{1}{2}y^2\right]-z=P \tag{2.4}$$

$$P_z+yP=\beta_n-1 \tag{2.5}$$

Note that substitution Eq. (2.4) into Eq. (2.5) gives Eq. (2.1).

We have from Eq. (2.5).

$$y=\frac{\beta_n-1}{P}-\frac{P_z}{P} \tag{2.6}$$

Substitution (2.6) into Eq. (2.4) leads to the equation

$$H_n\left[\frac{P_z^2}{2P^2}-\frac{P_{zz}}{P}-\frac{(\beta_n-1)^2}{2P^2}\right]=z+P \tag{2.7}$$

The set of Eqs. (2.4) and (2.5) can be considered as the Bäcklund transformation between Eqs. (2.1) and (2.7). Solution $P(z,\gamma_n)$ of Eq. (2.7) depends on $\gamma_n=(\beta_n-1)^2$ but there are two solutions $y(z,\beta_n)$ and $y(z,2-\beta_n)$ of Eq. (2.1) to obtain $P(z,\gamma_n)$. This fact shows how to get Eq. (2.2) from Eq. (2.6).

From Eq. (2.2) we have the Bäcklund transformation for solutions of Eq. (2.1) in the form

$$y(z,2-\beta_n)=y(z,\beta_n)+\frac{2\beta_n-2}{z-H_n\left[y_z-\frac{1}{2}y^2\right]} \tag{2.8}$$

Let us note the transformation (2.8) can be written for any operator

$$B_n\left[y_z-\tfrac{1}{2}y^2\right]$$

because we did not use the definition of the operator H_n given by formulas (1.6). However the transformation (2.8) allows to find the only one solution corresponding to any one known solution of Eq. (2.1). The reason of this fact is that the Eq. (2.1) is not solvable in general case.

Now it needs to prove another theorem.

Theorem 2. *Let $v(z,\beta_n)$ and $v(z,-1-\beta_n)$ be solutions of Eq. (2.1); then there is an invariant in the form*

$$I_2=v(z,\beta_n)-\frac{\beta_n}{Q}=v(z,-1-\beta_n)+\frac{1+\beta_n}{Q} \tag{2.9}$$

where

$$Q=G_n\left[-2v_z-2v^2\right]-z \tag{2.10}$$

and the operator G_n is determined by formulas

$$G_{n+2}=J_2[\omega]\,\theta_1[\omega]\,G_n,\quad G_0=1,$$

$$G_1[\omega]=\omega_{zz}+\tfrac{1}{4}\omega^2 \tag{2.11}$$

$$J_2=D^3+\tfrac{1}{2}D^2\omega D^{-1}+\tfrac{1}{2}D^{-1}\omega D^2+\tfrac{1}{8}(\omega^2D^{-1}+D^{-1}\omega^2),$$

$$D=\frac{d}{dz},\quad D^{-1}=\int dz \tag{2.12}$$

Proof. Let us note the hierarchy (2.1) also can be written in the form

$$\left(\frac{d}{dz}-2v\right)G_n\left[-2v_z-2v^2\right]+2zv+2\beta_n=0$$

$$(n=1,2,\dots) \tag{2.13}$$

where v is solution of Eq. (2.1) as well.

One can write Eq. (2.13) in the form of the following set of equations:

$$G_n\left[-2v_z-2v^2\right]-z=Q \tag{2.14}$$

$$\left(\frac{d}{dz}-2v\right)Q+2\beta_n+1=0 \tag{2.15}$$

We get from Eq. (2.15)

$$v=\frac{Q_z}{Q}+\frac{2\beta_n+1}{2Q} \tag{2.16}$$

Substitution (2.16) into Eq. (2.14) leads to the following equation:

$$G_n\left[\frac{Q_z^2}{2Q^2}-\frac{Q_{zz}}{2Q}-\frac{(2\beta_n+1)^2}{2Q^2}\right]=z+Q \tag{2.17}$$

From Eq. (2.17) we can see there is the solution $Q(z,\delta_n)$ where $\delta_n=(2\beta_n+1)^2$. This solution corresponds to $v(z,\beta_n)$ and $v(z,-1-\beta_n)$. This fact leads to Eq. (2.9).

Substitution (2.10) into Eq. (2.9) gives the Bäcklund transformation for solutions of Eq. (2.1) in the form

$$v(z,-1-\beta_n)=v(z,\beta_n)+\frac{1+2\beta_n}{z-G_n\left[-2v_z-2v^2\right]} \tag{2.18}$$

Definition 1. *Let transformations (2.8) and (2.18) be called the double Bäcklund transformations for the solutions of Eq. (2.1).*

At $n=1$ transformations (2.8) and (2.18) were obtained in [19], but at n ≥ 2 they were not found previously.

Transformations (2.8) and (2.18) allow to find the rational and the special solutions of Eq. (2.1). At the beginning we take the trivial solution of Eq. (2.1). Then we find solution taking into account transformations (2.8) and (2.18). After that we use these solutions in formulas (2.18) and (2.8) to obtain new solutions. This algorithm may be further continued. One can find a lot of solutions using this way. The existence of infinite number the rational or the special solutions for equation studied is known as a criterion of the equation integrability. The solutions of Eq. (2.1) can be obtained using the double Bäcklund transformations. They are consequences of two discrete symmetries of original equation. As a result ODEs will be solvable if they admit the double Bäcklund transformations.

3. Application of the Painlevé truncation approach to the K_{II} hierarchy

Let us show there is a connection between the double Bäcklund transformations and the Painlevé truncation approach [20,21].

The application of this approach is known to obtain the Bäcklund transformations, the Lax pairs, the Darboux transformations and other characteristics of nonlinear partial differential equations [6,20–27]. It was shown [12,28] how to use this approach to find the Bäcklund transformations for ODEs. Authors suggested the algorithm to find the Bäcklund transformations of the P_{II} hierarchy, third and fourth Painlevé equations.

Later we are going to demonstrate the application of the Painlevé truncation approach to obtain the double Bäcklund transformations for the solutions of the K_{II} hierarchy.

Taking into account the Painlevé truncation approach we can find that solutions of Eq. (2.1) can be presented in the form of two truncated approach

$$y=-\frac{2\varphi_z}{\varphi}+\frac{\varphi_{zz}}{\varphi_z} \tag{3.1}$$

and

$$v=\frac{\Psi_z}{\Psi}-\frac{\Psi_{zz}}{2\Psi_z} \tag{3.2}$$

Without loss of generality let us use transformations

$$y=\frac{\varphi_{zz}}{\varphi_z} \tag{3.3}$$

and

$$v=-\frac{\Psi_{zz}}{2\Psi_z} \tag{3.4}$$

because transformations $\varphi\to-1/\varphi$ in (3.3) and $\Psi\to-1/\Psi$ in (3.4) lead to Eqs. (3.1) and (3.2) [12].

Assuming (3.3) in Eq. (2.1) we obtain relations in the form

$$E_n = \left(\frac{d}{dz} + y\right) H_n\left[y_z - \tfrac{1}{2}y^2\right] - zy - \beta_n$$
$$= \left(\frac{d}{dz} + y\right)\left(H_n\left[y_z - \tfrac{1}{2}y^2\right] - z - (\beta_n - 1)\frac{\varphi}{\varphi_z}\right)$$
$$= \left(\frac{d}{dz} + y\right)\left(H_n\left[\tfrac{1}{2}\{ \varphi ; z\}\right] - z - (\beta_n - 1)\frac{\varphi}{\varphi_z}\right) \tag{3.5}$$

where $\{\varphi;z\}$ is the Schwarzian derivative

$$\{\varphi;z\} = \frac{\varphi_{zzz}}{\varphi_z} - \frac{3}{2}\frac{\varphi_{zz}^2}{\varphi_z^2} \tag{3.6}$$

Eq. (3.5) were obtained taking into account the identity

$$y_z - \tfrac{1}{2}y^2 = \tfrac{1}{2}\{\varphi;z\} \tag{3.7}$$

From Eq. (3.5) we have

$$H_n\left[\tfrac{1}{2}\{\varphi;z\}\right] - z - (\beta_n - 1)\frac{\varphi}{\varphi_z} = 0 \tag{3.8}$$

This one is the singular manifold equation corresponding to Eq. (2.1).

On the other hand we know Eq. (2.1) can be presented in the form (2.13).

Assuming (3.4) in Eq. (2.13) we obtain relations

$$E_n = -\tfrac{1}{2}\left(\frac{d}{dz} - 2v\right) G_n\left[-2v_z - 2v^2\right] - zv - \tilde{\beta}_n$$
$$= -\tfrac{1}{2}\left(\frac{d}{dz} - 2v\right)\left(G_n\left[-2v_z - 2v^2\right] - z + \left(2\tilde{\beta}_n + 1\right)\frac{\Psi}{\Psi_z}\right) = -\tfrac{1}{2}\left(\frac{d}{dz} - 2v\right)$$
$$\times\left(G_n[\{\Psi;z\}] - z + \left(2\tilde{\beta}_n + 1\right)\frac{\Psi}{\Psi_z}\right) \tag{3.9}$$

We found Eq. (3.9) taking into account the identity

$$\{\Psi;z\} = -2v_z - 2v^2 \tag{3.10}$$

Equation

$$G_n[\{\Psi;z\}] - z + \left(2\tilde{\beta}_n + 1\right)\frac{\Psi}{\Psi_z} = 0 \tag{3.11}$$

is the second singular manifold equation corresponding to Eq. (2.1).

Now let us consider Eqs. (3.8) and (3.11). It was shown [15] solutions of Eq. (3.11) can be found at known solutions $\varphi(z,\beta_n)$ of Eq. (3.8) by the iterative formula [6]

$$\Psi_z = \frac{\varphi^4}{\varphi_z^2} \tag{3.12}$$

where $\Psi = \Psi(z, 2 - \beta_n)$.

Let us note that Eq. (3.12) can be obtained from the equality

$$\frac{\Psi_{zz}}{2\Psi_z} = \frac{2\varphi_z}{\varphi} - \frac{\varphi_{zz}}{\varphi_z} \tag{3.13}$$

after integration (3.13) over z.

Taking into account Eqs. (3.3) and (3.4) we have from Eq. (3.13)

$$\frac{\varphi_z}{\varphi} = \tfrac{1}{2}\left[y(z,\beta_n) - y(z, 2 - \beta_n)\right] \tag{3.14}$$

Substitutions (3.7) and (3.14) into (3.8) lead to the following equation:

$$I_n^{(1)} = H_n\left[y_z - \tfrac{1}{2}y^2\right]$$
$$- z - \frac{2\beta_n - 2}{y(z,\beta_n) - y(z, 2 - \beta_n)} = 0 \tag{3.15}$$

We need to formulate the following lemma:

Lemma 1. *Let $y(z,\beta_n)$ and $y(z, 2-\beta_n)$ be solutions of hierarchy (2.1); then there is the equality*

$$y_z(z,\beta_n) - \tfrac{1}{2}y^2(z,\beta_n)$$
$$= y_z(z, 2 - \beta_n) - \tfrac{1}{2}y^2(z, 2 - \beta_n) \tag{3.16}$$

Proof. Replace $\beta_n \to 2 - \beta_n$ in (3.15). We get

$$H_n\left[y_z(z, 2 - \beta_n) - \tfrac{1}{2}y^2(z, 2 - \beta_n)\right]$$
$$= z + \frac{2 - 2\beta_n}{y(z, 2 - \beta_n) - y(z,\beta_n)} \tag{3.17}$$

Using (3.15) and (3.17) we have the equality (3.16).

Now we can prove the following theorem:

Theorem 3. *Let $I_n^{(1)}$ be the equation defined by formula (3.15); then $I_n^{(1)}$ is the special integral of hierarchy (2.1).*

Proof. Eq. (3.15) can be considered as the special integral of Eq. (2.1). To check this statement we have

$$\left(\frac{d}{dz}+y\right)I_n^{(1)}=\frac{2(\beta_n-1)}{\left(y(z,\beta_n)-y(z,2-\beta_n)\right)^2}$$
$$\left(y_z(z,\beta_n)-\tfrac{1}{2}y^2(z,\beta_n)-y_z(z,2-\beta_n)\right.$$
$$\left.+\tfrac{1}{2}y^2(z,2-\beta_n)\right)=0 \tag{3.18}$$

It is easy to obtain the transformation (2.8) from Eq. (3.15). Therefore the singular manifold Eq. (3.8) leads to the one of the double Bäcklund transformations.

However solutions of Eq. (3.8) can be obtained at known solutions of Eq. (3.11) by formula [6,15]

$$\varphi_z=\frac{\Psi}{\Psi_z^{1/2}} \tag{3.19}$$

where $\Psi=\Psi\left(z,\tilde{\beta}_n\right)$ and $\varphi=\varphi\left(z,-1-\tilde{\beta}_n\right)$.

Let us note that Eq. (3.19) can be found from equality

$$\frac{\varphi_{zz}}{\varphi_z}=\frac{\Psi_z}{\Psi}-\frac{\Psi_{zz}}{2\Psi_z} \tag{3.20}$$

after integration (3.20) over z.

Using (3.3) and (3.4) we obtain from Eq. (3.20)

$$\frac{\Psi_z}{\Psi}=v\left(z,-1-\tilde{\beta}_n\right)-v\left(z,\tilde{\beta}_n\right) \tag{3.21}$$

Substitutions (3.10) and (3.21) into (3.11) give the following equation:

$$I_n^{(2)}=G_n\left[-2v_z-2v^2\right]$$
$$-z-\frac{2\tilde{\beta}_n+1}{v\left(z,\tilde{\beta}_n\right)-v\left(z,-1-\tilde{\beta}_n\right)}=0 \tag{3.22}$$

We need to prove the following lemma:

Lemma 2. *Let $v\left(z,\tilde{\beta}_n\right)$ and $v\left(z,-1-\tilde{\beta}_n\right)$ be solutions of the hierarchy (2.1); then there is the equality*

$$\nu_z\left(z,\tilde{\beta}_n\right)+v^2\left(z,\tilde{\beta}_n\right)=v_z\left(z,-1-\tilde{\beta}_n\right)$$
$$+v^2\left(z,-1-\tilde{\beta}_n\right) \tag{3.23}$$

Proof. Replace $\tilde{\beta}_n\to-1-\tilde{\beta}_n$ in (3.22). We have

$$G_n\left[-2\nu_z\left(z,-1-\tilde{\beta}_n\right)-2v^2\left(z,-1-\tilde{\beta}_n\right)\right]$$
$$=z+\frac{2\tilde{\beta}_n+1}{v\left(z,\tilde{\beta}_n\right)-v\left(z,-1-\tilde{\beta}_n\right)}=0 \tag{3.24}$$

Taking into account formulas (3.22) and (3.24) we find the equality (3.23).

Now the following theorem can be proved:

Theorem 4. *Let $I_n^{(2)}$ be the equation defined by formula (3.22); then $I_n^{(2)}$ is the special integral of hierarchy (2.1).*

Proof. Eq. (3.22) also can be considered as the special integral of Eq. (2.1). Let us prove this statement using direct calculations. We have the equality

$$\left(\frac{d}{dz}-2\nu\right)I_n^{(2)}=\frac{2\tilde{\beta}_n+1}{\left(v\left(z,\tilde{\beta}_n\right)-v\left(z,-1-\tilde{\beta}_n\right)\right)^2}$$
$$\left[\nu_z\left(z,\tilde{\beta}_n\right)+v^2\left(z,\tilde{\beta}_n\right)-\nu_z\left(z,-1-\tilde{\beta}_n\right)\right.$$
$$\left.-v^2\left(z,-1-\tilde{\beta}_n\right)\right]=0 \tag{3.25}$$

taking into account Eq. (3.23).

We obtained that singular manifold Eq. (3.8) and (3.11) are transformed to the special integrals of Eq. (2.1). The double Bäcklund transformations (2.8) and (2.18) are found from the special integrals (3.15) and (3.22).

4. Rational solutions of the K_{II} hierarchy

Several rational solutions of Eq. (2.1) at $n=1$ were obtained in [15,19]. To find them in [15] solutions of the singular manifold Eqs. (3.8) and (3.11)were used. However we needed to integrate several expressions using this approach. And because of this integration a lot of rational solutions can not be found.

Formulas (2.8) and (2.18) eliminate this drawback.

Let us demonstrate the application of the double Bäcklund transformation to find rational solutions of Eq. (2.1) at $n=1$. In this case we have Eq. (2.1) in the form [15,19]

$$y_{zzzz}+5y_z y_{zz}-5yy_z^2-5y^2y_{zz}+y^5-zy-\beta_1=0 \tag{4.1}$$

The double Bäcklund transformations for solutions of Eq. (4.1) take the following form

$$y(z,2-\beta_1)=y(z,\beta_1)+\frac{2\beta_1-2}{z-y_{zzz}+yy_{zz}-3y_z^2+4y^2y_z-y^4} \tag{4.2}$$

and

$$v(z,-1-\beta_1)=v(z,\beta_1)+\frac{2\beta_1+1}{z+2v_{zzz}+4vv_{zz}+3v_z^2-2v^2v_z-v^4} \tag{4.3}$$

Let us take the trivial solution of Eq. (4.1) $y(z,0)=v(z,0)=0$ at $\beta_1=0$. By formulas (4.2) and (4.3) we have

$$v(z,-1)=\frac{1}{z},\quad y(z,2)=-\frac{2}{z} \tag{4.4}$$

Taking into account solutions (4.4) we obtain

$$y(z,3)=-\frac{3}{z},\quad v(z,-3)=\frac{3(z^5-24)}{z(z^5+36)} \tag{4.5}$$

from transformations (4.2) and (4.3).

Using rational solutions (4.5) we obtain

$$v(z,-4)=\frac{4}{z},\quad y(z,5)=-\frac{5z^4(z^5+216)}{z^{10}-108z^5-5184} \tag{4.6}$$

and

$$y(z,6)=-\frac{6(z^5+336)}{z(z^5-504)},$$
$$v(z,-6)=\frac{6(z^{20}-576z^{15}-912384z^{10}-459841536z^5-3153199104)}{z(z^5-144)(z^{15}-1152z^{10}+1824768z^5+131383296)} \tag{4.7}$$

Now let us find the rational solutions of Eq. (2.1) at $n=2$. In this case from Eq. (2.1) we have the following equation

$$\begin{aligned}&y_{zzzzzz}+7y_z y_{zzzz}-28y_z^2y_{zz}-21yy_{zz}^2+14y_{zz}y_{zzz}\\&-7y^2y_{zzzz}-\tfrac{28}{3}yy_z^3+14y_{zz}y^4+28y_z^2y^3\\&-28yy_zy_{zzz}-14y_zy_{zz}y^2-\tfrac{4}{3}y^7-zy-\beta_2=0\end{aligned} \tag{4.8}$$

Rational solutions of Eq. (4.8) can be found using formulas (2.8) and (2.18) at $n=2$ if we assume $y(z,0)=v(z,0)=0$ at $\beta_2=0$.

We find

$$v(z,-1)=\frac{1}{z},\quad v(z,2)=-\frac{2}{z},\quad v(z,3)=-\frac{3}{z},\quad v(z,-3)=\frac{3}{z},\quad v(z,-4)=\frac{4(z^7+1296)}{z(z^7-1728)},$$

$$v(z,5)=-\frac{5}{z},\quad y(z,6)=-\frac{6(z^{14}-9504z^7+1244160)}{z(z^{14}+2592z^7-7464960)},\quad v(z,-6)=\frac{6}{z},$$

$$v(z,-7)=\frac{7z^6(z^{21}+12960z^{14}+2199360000z^7-15721205760000)}{z^{28}+47520z^{21}-2239488000z^{14}-7860602880000z^7+11319268147200000},$$

$$y(z,8)=-\frac{8(z^7-71280)}{z(z^7+95040)} \tag{4.9}$$

The rational solutions (4.4)–(4.7) were obtained in [15,19]. Solutions (4.9) are found in the frame of this work. The special solutions of Eq. (2.1) also can be found using formulas (2.8) and (2.18).

5. Conclusion

The double Bäcklund transformations for the solutions of the K_{II} hierarchy were found using two approaches. One of them is the application of discrete symmetry for two solutions of the K_{II} hierarchy. Another method was based on the application of the Painlevé truncation approach to the ordinary differential equations. Simultaneously the special integrals for the K_{II} hierarchy were obtained. As a consequence of these special integrals the double Bäcklund transformations were obtained. Taking into account the double Bäcklund transformations several rational solutions of the K_{II} hierarchy at $n=1$ and $n=2$ were found

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